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Solution of a tridiagonal operator equation

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Abstract

Let H be a separable Hilbert space with an orthonormal basis $\{e_n/n \in \mathbb{N}\}$, T be a bounded tridiagonal operator on H and T_n be its truncation on span $(\{e_1, e_2, \ldots, e_n\})$. We study the operator equation $Tx = y$ through its finite dimensional truncations $T_n x^n = y_n$. It is shown that if $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$ are bounded, then T is invertible and the solution of $Tx = y$ can be obtained as a limit in the norm topology of the solutions of its finite dimensional truncations. This leads to uniform boundedness of the sequence $\{T_n^{-1}\}$. We also give sufficient conditions for the boundedness of $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$ in terms of the entries of the matrix of T.

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1. Introduction

The numerical solution of ordinary and partial differential equations frequently leads to linear systems of equations involving matrices whose elements are zero except in a band sorrounding the main diagonal. For details regarding linear system arising from partial differential equations, we refer to [3]. One of the special type is a tridiagonal matrix. Application of Finite Differences or Finite Element methods to solve boundary value problems in one variable results in systems

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of equations whose matrices are banded and in case of some important examples these matrices turn out to be tridiagonal. There exist many well developed methods and efficient algorithms in the literature for solving these matrix equations or finding eigen values of these matrices.

We use the following notations throughout the paper. Let H denote a separable Hilbert space with an orthonormal basis $\{e_n/n \in \mathbb{N}\}$. Let H_n denote the linear span of $\{e_1, e_2, \ldots, e_n\}$, P_n, the orthogonal projection of H onto H_n. For $T \in \mathscr{B}(H)$, the class of bounded linear operators on H. $T_n = P_nT|_{H_n}$ and $Sp(T)$ will denote the spectrum of T. The operators $\{T_n\}$ are known as finite sections or Galerkin approximations of T. Matrix of $\{T_n\}$ (with respect to the basis $\{e_j/j \in \mathbb{N}\}$) consists of first n rows and n columns of the matrix T. For $x = \sum_j \alpha_j e_j$, x_n will denote $P_n(x)$. Thus $x_n = \sum_{j=1}^n \alpha_j e_j$.

Our interest is to study the solution of the operator equation $Tx = y$ where T is an infinite tridiagonal matrix which can be regarded as a bounded operator on H. In particular, we wish to first answer the question of invertibility of a tridiagonal operator and then obtain the solution of the operator equation $Tx = y$. For this, we consider the finite dimensional approximations T_n of T. By assuming that each T_n is invertible along with certain other conditions, we show that T is invertible. The result is contained in Theorem 5.1. In the next step, we try to find certain conditions so that the hypothesis of Theorem 5.1 could be verified for a given operator. It turns out that the required conditions can be stated in terms of the entries of the matrices T_n.

We organize the paper in the following way. In Section 2, we provide some mathematical and historical background. In Section 3, we record certain nice properties of the tridiagonal operator. In Section 4, we obtain some consequences of the boundedness of $\{\|T_n^{*^{-1}}(e_n)\|\}$. These are used in Section 5 to prove the main result (Theorem 5.1). In Section 6, we discuss some verifiable criterions and illustrate these with some examples.

Unlike in the case of one variable, the application of Finite Differences or Finite Element methods to partial differential equations lead to matrices that are no longer banded. But these matrices are banded in a different sense. For example, when these methods are applied to two dimensional Laplace equation, the resulting matrices can be viewed as tridiagonal matrices whose entries are tridiagonal matrices (see [9] for details). At present it is not clear whether our methods can be applied to this class of problems. This is an interesting question to be investigated in future.

2. Background

Though it may seem natural to expect that the behaviour of T can be predicted from the behaviour of T_n for large values of n, it is well known that this expectation is false unless some additional assumptions are made about T_n and/or T. For example, in general the invertibility of T_n for all n does not imply the invertibility of T. Consider for example $T : \ell^2 \mapsto \ell^2$ defined by

$$Tx = \left(\alpha_1, \frac{\alpha_2}{2}, \ldots, \frac{\alpha_n}{n}, \ldots\right), \quad (\alpha_1, \alpha_2, \ldots) \in \ell^2.$$

Then each T_n is invertible. In fact,

$$T_n^{-1}(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1, 2\alpha_2, \ldots, n\alpha_n).$$

But T is not invertible as $Sp(T) = \{0\} \cup \{1/n, n \in \mathbb{N}\}$. Notice that $\|T_n^{-1}\| = n = \|T_n^{-1}e_n\|$. Similarly diagonal dominance property of a finite matrix implies its invertibility whereas the above example illustrates that this is not the case with an infinite matrix. For the study of diagonal dominance property and finite dimensional matrices we refer to [7]. However, in the case of an infinite matrix it can be proved that if $T \in \mathscr{B}(H)$ has a strict row and column dominance property, that is if for some $\epsilon > 0$, $|\alpha_{jj}| > \sum_{i \neq j} |\alpha_{ij}| + \epsilon$ for all j and $|\alpha_{ii}| > \sum_{j \neq i} |\alpha_{ij}| + \epsilon$ for all i, then

T is invertible. We can deduce from this an infinite dimensional version of Gerschgorin theorem. Let $T = (\alpha_{ij})$, $r_i = \sum_{j \neq i} |\alpha_{ij}|$, $r_i' = \sum_{k \neq i} |\alpha_{ki}|$. If $D_i = B(\alpha_{ii}, r_i)$ and $D_i' = B(\alpha_{ii}, r_i')$, then $Sp(T) \subset \overline{\left(\bigcup_i D_i \cup D_i'\right)}$. Further it is also known that if there exists an n_0 such that T_n is invertible for all $n \geqslant n_0$, then the solutions x^n of $T_n x^n = y_n$ lead to the solution x of $Tx = y$ where $x = \lim_n x^n$ if and only if T is invertible. The invertibility of an operator T and the invertibility of its finite dimensional truncations T_n are discussed in detail for Toeplitz operators in [6]. The problem of computing spectrum $Sp(T)$ through its finite dimensional truncations under certain assumptions are discussed in [1,2]. Finally, it may be noted that several of tridiagonal operators are not invertible. Prominent among these are certain class of almost Mathieu operators and, by consistency, any discretization by Finite Differences of differential equations (see [4,8] for details).

In this paper, we try to answer the question of finding conditions under which a tridiagonal operator equation $Tx = y$ has a solution, when each of the corresponding finite dimensional truncations $T_n x^n = y_n$ has a solution. Further such conditions prove that $\{T_n^{-1}\}$ is uniformly bounded.

In practice, to apply these sort of theorems to concrete cases, we must have easily verifiable conditions. For example, if the conditions can be stated in terms of the entries of the matrices, then it serves the purpose. The best known illustration of such a condition is the diagonal dominance property mentioned above. Suppose the tridiagonal operator is such that its off-diagonal elements are 1 and product of k diagonal elements in absolute value is greater than 2^k (instead of assuming that absolute value of each diagonal element to be greater than 2), then the operator T is invertible (see [5]). This essentially tells us that even if one of the diagonal elements is very small but the product is large enough, then we obtain the invertibility of T. In this paper we prove a still weaker condition. For example, when $k = 2$, and again if we assume that T to be a tridiagonal operator with off-diagonal elements 1, then we prove that the condition

$$(|d_i||d_{i+1}| - 2)(|d_{i-1}||d_i| - 2) \geqslant 4 + \epsilon$$

gives the sufficient conditions for the hypothesis of Theorem 5.1 to be satisfied. Similarly for $k = 3$, if the sequence $\{d_i\}$ satisfies the condition

$$\begin{aligned} |d_i|(|d_{i+1}||d_{i+2}|-1)(|d_{i-1}||d_{i-2}|-1) \geqslant (|d_{i-1}||d_{i-2}|-1)|d_{i+2}| + (|d_{i+1}||d_{i+2}|-1)|d_{i-2}| \\ +(|d_{i+1}||d_{i+2}| - 1) + (1+\epsilon)(|d_{i-1}||d_{i-2}| - 1) \end{aligned}$$

and $(|d_i||d_{i+1}| - 1) \geqslant \eta$ for all i where $\eta > 0$ with $|\det T_3| \geqslant (1+\epsilon)$, then $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$ are bounded.

3. Properties of a tridiagonal operator

Let T be the tridiagonal operator defined by

$$Te_1 = d_1 e_1 + u_2 e_2$$

and

$$Te_n = c_{n-1} e_{n-1} + d_n e_n + u_{n+1} e_{n+1} \quad \text{for } n \geqslant 2,$$

where $\{c_n\}$, $\{d_n\}$ and $\{u_n\}$ are bounded sequences of complex numbers. For the sake of notational convenience, we give the proof when c_n, u_n, d_n are all real. We can modify this in an obvious manner if c_n, d_n, u_n are complex numbers. One of the interesting characteristics of the tridiagonal operator T is the recurrence relation concerning the determinants of its finite sections T_n, namely

$$\det T_n = d_n \det T_{n-1} - u_n c_{n-1} \det T_{n-2}. \tag{1}$$

Relation (1) can be easily verified by expanding the matrix of T_n along the last row.

The tridiagonal nature of T helps us to expand $T_n^{-1} e_n$ and $T_n^{*^{-1}}(e_n)$ in terms of determinants of $T_1, T_2, \ldots, T_n$. In fact, if we consider the matrix form of T_n^*, then $T_n^{*^{-1}}(e_n) = (\det T_n)^{-1}[k_1 e_1 + k_2 e_2 + \cdots + k_n e_n]$ where $k_1, k_2, \ldots, k_n$ are the cofactors of the elements in the last row respectively. The later equality follows from the fact that for a given matrix, $A = (a_{ij})_{n\times n}$,

$$\sum_{i=1}^{n} a_{ij} A_{ri} = \delta_{jr}(\det A),$$

where $A_{r,i}$ denotes the cofactor of $a_{r,i}$.

The tridiagonal form of T_n simplifies our task of computing the cofactors of the elements of the last row. k_m is the cofactor of the element in the place (n, m) of T_n^* where (n, m) denotes the nth row and mth column. It is nothing but $(-1)^{n+m}$ multiplied by the determinant of T_n^* with nth row and mth column deletion. If we now expand this determinant along the last column, there is only one non-zero element in the last column namely u_n, then expanding along the last column the only non-zero element in the last column under consideration is u_{n-1}. Repeating this process upto $(m+1)$th column (in the original determinant), we get that $u_n u_{n-1} \cdots u_{m+1}$. Because of mth column deletion, the left out determinant will be precisely $\det T_{m-1}^* = \det T_{m-1}$. Thus $k_m = (-1)^{m+n} u_{m+1} u_{m+2} \cdots u_{n-1} u_n\ (\det T_{m-1})$. The above technique can be understood by the following simple example:

$$T_4 = \begin{pmatrix} d_1 & c_1 & 0 & 0 \\ u_2 & d_2 & c_2 & 0 \\ 0 & u_3 & d_3 & c_3 \\ 0 & 0 & u_4 & d_4 \end{pmatrix},$$

$$T_4^* = \begin{pmatrix} d_1 & u_2 & 0 & 0 \\ c_1 & d_2 & u_3 & 0 \\ 0 & c_2 & d_3 & u_4 \\ 0 & 0 & c_3 & d_4 \end{pmatrix},$$

$$\begin{aligned} T_4^{*^{-1}}(e_4) = (\det T_4)^{-1}\big[&(\text{cofactor } 0\ (4,1))\ e_1 + (\text{cofactor } 0\ (4,2))\ e_2 \\ &+(\text{cofactor } c_3)\ e_3 + (\text{cofactor } d_4)\ e_4\big]. \end{aligned}$$

Hence

$$\begin{aligned} (\det T_4) T_4^{*^{-1}}(e_4) &= -\begin{vmatrix} u_2 & 0 & 0 \\ d_2 & u_3 & 0 \\ c_2 & d_3 & u_4 \end{vmatrix} e_1 + \begin{vmatrix} d_1 & 0 & 0 \\ c_1 & u_3 & 0 \\ 0 & d_3 & u_4 \end{vmatrix} e_2 \\ &\quad - \begin{vmatrix} d_1 & u_2 & 0 \\ c_1 & d_2 & 0 \\ 0 & c_2 & u_4 \end{vmatrix} e_3 + \begin{vmatrix} d_1 & u_2 & 0 \\ c_1 & d_2 & u_3 \\ 0 & c_2 & d_3 \end{vmatrix} e_4 \\ &= -u_2 u_3 u_4 e_1 + u_3 u_4 (\det T_1) e_2 - u_4 (\det T_2) e_3 + (\det T_3) e_4. \end{aligned}$$

Thus we get

$$\begin{aligned} T_n^{*^{-1}}(e_n) = (-1)^{n+1}(\det T_n)^{-1}\big[&u_2 u_3 \cdots u_n e_1 - u_3 u_4 \cdots u_n (\det T_1) e_2 \\ &+ \cdots + (-1)^{n-2} u_n (\det T_{n-2}) e_{n-1} + (-1)^{n-1} (\det T_{n-1}) e_n\big]. \end{aligned} \tag{2}$$

Similarly it can be shown that

$$T_n^{-1}(e_n) = (-1)^{n+1}(\det T_n)^{-1}\big[c_1c_2\cdots c_{n-1}e_1 - c_2\cdots c_{n-1} \times (\det T_1)e_2 + \cdots + (-1)^{n-1}(\det T_{n-1})e_n\big]. \tag{3}$$

If

$$x_n = \alpha_1^{(n)}e_1 + \alpha_2^{(n)}e_2 + \cdots + \alpha_n^{(n)}e_n,$$

then $T(x_n)$ and $T_n(x_n)$ differs only in the last component, namely

$$T(x_n) = T_n(x_n) + \alpha_n^{(n)}u_{n+1}e_{n+1}. \tag{4}$$

Further if $x = \sum \alpha_i e_i$, then it can be easily shown that

$$P_nT(x) = T_n(x_n) + \alpha_{n+1}c_ne_n.$$

4. Boundedness of $\{\|T_n^{*^{-1}}e_n\|\}$

Assume that $\{\|T_n^{*^{-1}}e_n\|\}$ is bounded by a number say L. If we let

$$x^n = \alpha_1^{(n)}e_1 + \alpha_2^{(n)}e_2 + \cdots + \alpha_n^{(n)}e_n,$$

then we will show that $\{u_{n+1}\alpha_n^{(n)}\} \subset \ell^2$. The condition $\|T_n^{*^{-1}}e_n\| \leqslant L$ is equivalent to

$$u_2^2u_3^2\cdots u_n^2 + u_3^2u_4^2\cdots u_n^2(\det T_1)^2 + \cdots + u_n^2(\det T_{n-2})^2 + (\det T_{n-1})^2 \leqslant L^2(\det T_n)^2. \tag{5}$$

Define

$$a_i = \frac{\det T_i}{u_2\cdots u_{i+1}}.$$

Then (3) can be rewritten as

$$1 + a_1^2 + \cdots + a_{n-1}^2 \leqslant L^2u_{n+1}^2a_n^2, \tag{6}$$

or using the boundedness of $\{u_n\}$, we can write

$$1 + a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \leqslant B^2a_n^2. \tag{7}$$

But

$$\begin{aligned}\alpha_n^{(n)} = \langle x^n, e_n\rangle &= \langle T_n^{-1}(y_n), e_n\rangle \\ &= \sum_{i=1}^{n}\beta_i\langle T_n^{-1}e_i, e_n\rangle \\ &= \sum_{i=1}^{n}\beta_i\langle e_i, T_n^{*-1}e_n\rangle. \\ &= (\det T_n)^{-1}\sum_{i=1}^{n}(-1)^{i+n}\beta_iu_{i+1}\cdots u_n\det T_{i-1}\end{aligned}$$

$$= (\det T_n)^{-1} \sum_{i=1}^{n} (-1)^{i+n} \beta_i u_2 \cdots u_n a_{i-1}$$
$$= (\det T_n)^{-1} u_2 \cdots u_n \sum_{i=1}^{n} (-1)^{i+n} \beta_i a_{i-1}.$$

Now, in order to show that $\{u_{n+1}\alpha_n^{(n)}\} \subset \ell^2$, we make use of certain estimates.

Lemma 4.1

$$1^r + 2^r + \cdots + n^r \geqslant \frac{n^{r+1}}{r+1} \geqslant \frac{(n+1)^{r+1}}{2^{r+1}(r+1)}.$$

Proof

$$\frac{j^{r+1} - (j-1)^{r+1}}{r+1} = \int_{j-1}^{j} u^r \, \mathrm{d}u \leqslant j^r \int_{j-1}^{j} \mathrm{d}u = j^r.$$

Summing for $j = 1, 2, \ldots, n$, we obtain the result. □

Lemma 4.2. *For any non-negative integer r and $n \geqslant 1$, we have*

$$a_{k+n}^2 \geqslant (1 + a_1^2 + \cdots + a_k^2) \frac{n^r}{2^{\frac{r(r+1)}{2}} r! B^{2(r+1)}}.$$

Here B is defined by Eq. (7).

Proof. When $r = 0$, the result will follow immediately from (7). Assume that the result is true for r. Then, by summing over $n = 1, 2, \ldots, (N-1)$, we get

$$\begin{aligned} a_{k+1}^2 + \cdots + a_{k+N-1}^2 &\geqslant \frac{(1 + a_1^2 + \cdots + a_k^2)}{2^{\frac{r(r+1)}{2}} r! B^{2(r+1)}} \sum_{n=1}^{N-1} n^r \\ &\geqslant \frac{(1 + a_1^2 + \cdots + a_k^2)}{2^{\frac{r(r+1)}{2}} r! B^{2(r+1)}} \frac{N^{r+1}}{2^{r+1}(r+1)} \quad \text{(by Lemma 4.1)} \\ &= \frac{(1 + a_1^2 + \cdots + a_k^2) N^{r+1}}{2^{\frac{(r+1)(r+2)}{2}} (r+1)! B^{2(r+1)}}. \end{aligned}$$

Thus

$$\begin{aligned} a_{k+N}^2 &\geqslant \frac{1}{B^2} \left[(1 + a_1^2 + \cdots + a_k^2) + (a_{k+1}^2 + \cdots + a_{k+N-1}^2) \right] \\ &\geqslant \frac{(1 + a_1^2 + \cdots + a_k^2) N^{r+1}}{2^{\frac{(r+1)(r+2)}{2}} (r+1)! B^{2(r+2)}} \end{aligned}$$

Hence the result is true for $r + 1$. □

Now we are in a position to prove the required result.

Proposition 4.3. *Let T be the tridiagonal operator defined by*

$$T e_n = c_{n-1} e_{n-1} + d_n e_n + u_{n+1} e_{n+1},$$

where $\{c_n\}$, $\{d_n\}$ and $\{u_n\}$ are bounded sequences of complex numbers. Suppose T_n is invertible for all n and there exists a constant $L > 0$ such that $\|T_n^{*^{-1}}(e_n)\| \leqslant L$ for all n, then $\{u_{n+1}\alpha_n^{(n)}\} \subset \ell^2$.

Proof. Recall

$$\alpha_n^{(n)} = (\det T_n)^{-1} u_2 \cdots u_n \sum_{i=1}^{n} (-1)^{i+n} \beta_i a_{i-1}$$
$$= \frac{1}{a_n u_{n+1}} \sum_{i=1}^{n} (-1)^{i+n} \beta_i a_{i-1}.$$

Therefore,

$$u_{n+1}\alpha_n^{(n)} = \frac{1}{a_n} \sum_{i=1}^{n} (-1)^{i+n} \beta_i a_{i-1}$$
$$|u_{n+1}\alpha_n^{(n)}| \leqslant \frac{1}{|a_n|} \sum_{i=1}^{n} |\beta_i||a_{i-1}|$$

By Lemma 4.2, we get

$$\frac{1}{|a_n|} \sum_{i=1}^{n} |\beta_i||a_{i-1}| \leqslant c \sum_{i=1}^{n} \frac{|\beta_i|}{(n-i+1)^4}$$

by taking $r = 8$. We break the sum for i as $i \leqslant \frac{n}{2}$ and $i > \frac{n}{2}$.

When $i \leqslant \frac{n}{2}$, we get the R.H.S. to $\mathrm{O}\left(\frac{1}{n^3}\right)$, which gives an ℓ^2 sequence. Now, let $M < n \leqslant N$. Then

$$|u_{n+1}\alpha_n^{(n)}| \leqslant c \sum_{\substack{i > M/2 \\ i \leqslant n}} \frac{|\beta_i|}{(n-i+1)^4} + \text{an } \ell^2 \text{ sequence.}$$

Consider

$$\sum_{\frac{M}{2} < n \leqslant N} \left| \sum_{i > \frac{M}{2}} \frac{|\beta_i|}{(n+1-i)^4} \right|^2 = \sum_{\frac{M}{2} < n} \sum_{\substack{i_1, i_2 \\ i_1, i_2 \leqslant n}} \frac{|\beta_{i_1}||\beta_{i_2}|}{(n+1-i_1)^4 (n+1-i_2)^4}$$
$$\leqslant \sum_{\substack{i_1, i_2 \\ i_1 > \frac{M}{2} \\ i_1 \leqslant i_2}} \sum_{n \geqslant i_2} \frac{|\beta_{i_1}|^2 + |\beta_{i_2}|^2}{(n+1-i_1)^4 (n+1-i_2)^4}.$$

Now,

$$\sum_{i_1, i_2, n} \frac{|\beta_{i_1}|^2}{(n+1-i_1)^4 (n+1-i_2)^4} \leqslant \sum_{N \geqslant i_1 > \frac{M}{2}} |\beta_{i_1}|^2 \sum_{i_2 \geqslant i_1} \frac{1}{(i_2+1-i_1)^4} \sum_{n \geqslant i_2} \frac{1}{(n+1-i_2)^4}$$
$$< \epsilon \quad \text{as } M, N \to \infty.$$

Similarly, we can show that

$$\sum_{i_1,i_2,n} \frac{|\beta_{i_2}|^2}{(n+1-i_1)^4(n+1-i_2)^4} < \epsilon \quad \text{as } M, N \to \infty,$$

thus proving our assertion. □

5. The main result

In this section we prove our main result.

Theorem 5.1. *Let T be the tridiagonal operator defined by*

$$Te_n = c_{n-1}e_{n-1} + d_n e_n + u_{n+1}e_{n+1}$$

where $\{c_n\}$, $\{d_n\}$ and $\{u_n\}$ are bounded sequences of complex numbers. Suppose T_n is invertible for all n and there exists a constant K such that $0 < K < \infty$ and $\|T_n^{-1}e_n\| \leqslant K$ for all n. If the operator equation $Tx = y$ has a solution (i.e. if $y \in R(T)$, with $R(T)$ being the range of T), then this solution can be obtained as the limit of the solutions x^n of the operator equation $T_n x^n = y_n|_{H_n}$ in the norm topology. In other words, $T_n^{-1}(y_n) \to x$. In particular, T is 1–1.

In addition if $c_n \neq 0$, $u_n \neq 0$ for all n and there exists $L > 0$ such that $\|T_n^{*^{-1}}(e_n)\| \leqslant L$ for all n, then T is onto and hence invertible.

Proof. Let $y \in R(T)$, $x = \sum_{i=1}^{\infty} \alpha_i e_i$, and $T(x) = y$. Then

$$\langle T_n(x_n), e_n\rangle = u_n\alpha_{n-1} + d_n\alpha_n$$

and

$$\langle T(x), e_n\rangle = u_n\alpha_{n-1} + d_n\alpha_n + c_n\alpha_{n+1}.$$

Thus $T_n(x_n) + \alpha_{n+1}c_n e_n = y_n$ which implies that

$$T_n^{-1}(y_n) = x_n + c_n\alpha_{n+1}T_n^{-1}(e_n).$$

As $n \to 0$, $x_n \to x$, $\alpha_n \to 0$ and $\{\|T_n^{-1}(e_n)\|\}$ is bounded, we see that $T_n^{-1}(y_n) \to x$. It follows that T is 1–1 (by taking $y = 0$).

In order to prove T is onto, let $y \in H$. Then $y = \sum_{i=1}^{\infty} \beta_i e_i$, $\sum |\beta_i|^2 < \infty$. In fact, $\sum_i |\beta_i|^2 = \|y\|^2$. Put $y_n = \sum_{i=1}^{n} \beta_i e_i$. Since each T_n is onto $\exists\, x^n \in H_n$ such that $T_n x^n = y_n$.

We can write

$$x^n = \alpha_1^{(n)}e_1 + \alpha_2^{(n)}e_2 + \cdots + \alpha_n^{(n)}e_n.$$

Then it follows from Proposition 4.3 that $\alpha_n^{(n)}u_{n+1} \to 0$ as $n \to \infty$. Further by (4), we have,

$$T_n(x^n) = T(x^n) - \alpha_n^{(n)}u_{n+1}e_{n+1}. \tag{8}$$

Now, if we show that $\{x^n\}$ is a Cauchy sequence in H then there will exist an $x \in H$ such that $x^n \to x$ in H, then by continuity $T(x^n) \to T(x)$ and in the limit $T(x^n)$ and $T_n(x^n)$ coincide by (1). But

$$y = \lim_n y_n = \lim_n T_n(x^n) = \lim_n T(x^n) = Tx,$$

thus showing that T is onto. Further this argument clearly shows that the operator equation $Tx = y$ could be solved by restricting y to each H_n, for, the invertibility of each T_n provides the solution x^n for $T_n x^n = y_n$ and the resulting x which is obtained as a limit (in the norm) of x^n turns out to be the solution of $Tx = y$. Thus it remains to show that $\{x^n\}$ is a Cauchy sequence in H, which is proved in Lemma 5.2. $\square$

Lemma 5.2. *$\{x^n\}$ is a Cauchy sequence in H.*

Proof. Consider

$$\begin{aligned} x^{n+1} - x^n &= T_{n+1}^{-1}(y_{n+1}) - T_{n+1}^{-1}T_{n+1}(x^n) \\ &= T_{n+1}^{-1}(y_{n+1}) - T_{n+1}^{-1}(y_n) - \alpha_n^{(n)}u_{n+1}T_{n+1}^{-1}(e_{n+1}) \\ &= \beta_{n+1}T_{n+1}^{-1}(e_{n+1}) - \alpha_n^{(n)}u_{n+1}T_{n+1}^{-1}(e_{n+1}). \end{aligned}$$

Then for $M > N$,

$$x^M - x^N = \sum_{N \leqslant n < M} (\beta_{n+1} - \alpha_n^{(n)}u_{n+1})T_{n+1}^{-1}(e_{n+1}).$$

By (3), we have

$$\begin{aligned} T_n^{-1}(e_n) = (-1)^{n+1}(\det T_n)^{-1}\big[&c_1c_2\cdots c_{n-1}e_1 - c_2\cdots c_{n-1} \\ &\times(\det T_1)e_2 + \cdots + (-1)^{n-1}(\det T_{n-1})e_n\big]. \end{aligned}$$

Let $b_i = \frac{\det T_i}{c_1c_2\cdots c_i}$. Then

$$T_{n+1}^{-1}(e_{n+1}) = (c_{n+1})^{-1}(b_{n+1})^{-1}\sum_{i=1}^{n+1}(-1)^{i+n}b_{i-1}e_i.$$

Let $\nu_n = \beta_{n+1} - \alpha_n^{(n)}u_{n+1}$. Then by Proposition 4.3, $\nu_n \in \ell^2$. Then

$$\begin{aligned} \|x^M - x^N\|^2 &= \sum_{i=1}^{\infty}\left|\sum_{\substack{N \leqslant n < M \\ n+1 \geqslant i}} \nu_n c_{n+1}^{-1}(b_{n+1})^{-1}\right|^2 b_{i-1}^2 \\ &= \sum_{i=1}^{\infty}\left[\sum_{n_1,n_2} |\nu_{n_1}||\nu_{n_2}||c_{n_1+1}^{-1}||b_{n_1+1}^{-1}||c_{n_2+1}^{-1}||b_{n_2+1}^{-1}|\right] b_{i-1}^2 \\ &\leqslant 2\sum_{\substack{n_1,n_2 \\ n_1 \leqslant n_2}} \frac{|\nu_{n_1}||\nu_{n_2}|}{|c_{n_1+1}||c_{n_2+1}||b_{n_1+1}||b_{n_2+1}|}\sum_{i \leqslant n_1+1} b_{i-1}^2 \end{aligned}$$

$$\begin{aligned} \|x^M - x^N\|^2 &\leqslant c\sum_{n_1,n_2} \frac{|\nu_{n_1}||\nu_{n_2}|c_{n_1+1}^2 b_{n_1+1}^2}{|c_{n_1+1}||c_{n_2+1}||b_{n_1+1}||b_{n_2+1}|} \\ &= c\sum_{n_1 \leqslant n_2} \frac{|\nu_{n_1}||\nu_{n_2}||c_{n_1+1}||b_{n_1+1}|}{|c_{n_2+1}||b_{n_2+1}|} \\ &\leqslant c|\nu_{n_1}|^2 + c\sum_{n_1 < n_2} \frac{|\nu_{n_1}||\nu_{n_2}||c_{n_1+1}||b_{n_1+1}|}{|b_{n_2}|} \\ &\leqslant c|\nu_{n_1}|^2 + c\sum_{n_1 < n_2} \frac{|\nu_{n_1}||\nu_{n_2}||b_{n_1+1}|}{|b_{n_2}|} \end{aligned}$$

as $\{c_n\}$ is bounded. Consider

$$\sum_{n_1<n_2} \frac{|\nu_{n_1}||\nu_{n_2}||b_{n_1+1}|}{|b_{n_2}|} \leqslant \sum_{n_1<n_2} \frac{|\nu_{n_1}|^2|b_{n_1+1}|}{|b_{n_2}|} + \sum_{n_1<n_2} \frac{|\nu_{n_2}|^2|b_{n_1+1}|}{|b_{n_2}|}$$
$$\leqslant |\nu_{n_1}|^2 + |\nu_{n_1+1}|^2 + \sum_{n_2>n_1+1} \frac{|\nu_{n_1}|^2|b_{n_1+1}|}{|b_{n_2}|}$$
$$+ \sum_{n_2>n_1+1} \frac{|\nu_{n_2}|^2|b_{n_1+1}|}{|b_{n_2}|}.$$

Here, $\{\nu_n\} \subset \ell^2$. Now, applying Lemma 4.2 for $\{b_i\}$ instead of $\{a_i\}$, and following the same technique in the proof of Proposition 4.3, we can show that $\|x^M - x^N\| < \epsilon$ as N, $M \to \infty$. $\square$

Corollary 5.3. *If T satisfies the hypothesis of Theorem* 5.1, *then* $\{\|T_n^{-1}\|\}$ *is uniformly bounded.*

Proof. Let $y \in H$. Then $P_n y \in H_n$ and by Theorem 5.1, $\{T_n^{-1}(P_n y)\}$ converges, which shows that there exists M_y such that $\|T_n^{-1} P_n y\| \leqslant M_y$. Then by uniform boundedness principle $\{\|T_n^{-1} P_n\|\}$ is uniformly bounded. But it can be easily shown that $\|T_n^{-1} P_n\| = \|T_n^{-1}\|$, thus proving our assertion. $\square$

6. A verifiable criterion

As mentioned earlier, we obtain some sufficient conditions in terms of the entries of the finite sections of the tridiagonal operator T so that $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*-1} e_n\|\}$ are bounded. In order to state the main theorem, we need the following notations.

Let $k \geqslant 2$ be a positive integer. Let $A_{j,n}$ denote the $j \times j$ matrix

$$\begin{pmatrix} d_{kn} & u_{kn} & \cdots & \cdots \\ c_{kn-1} & d_{kn-1} & u_{kn-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & c_{kn-(j-1)} & d_{kn-(j-1)} \end{pmatrix}$$

and $B_{j,n}$ denote the $j \times j$ matrix

$$B_{j,n} = \frac{1}{\prod_{2k-2}^{2k-(j+1)} c_{kn-l} \prod_{2k-3}^{2k-(j+2)} u_{kn-l}} \times \begin{pmatrix} d_{kn-(2k-2)} & c_{kn-(2k-2)} & 0 & \cdots \\ u_{kn-(2k-3)} & d_{kn-(2k-3)} & c_{kn-(2k-3)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & u_{kn-(2k-(j+1))} & d_{kn-(2k-(j+1))} \end{pmatrix}.$$

Let $A_{j,n}^{(1)}$, $B_{j,n}^{(1)}$ denote similar matrices where d_i, u_i, c_i are replaced by d_{i+1}, u_{i+1}, c_{i+1} (with the corresponding changes in other entries) respectively. In fact, let $A_{j,n}^{(r)}$, $B_{j,n}^{(r)}$ denote the matrix where the entries of $A_{j,n}$, $B_{j,n}$ are shifted to the right by r respectively.

Now we are in a position to state the required result.

Theorem 6.1. *Let T be a tridiagonal operator $Te_n = c_{n-1}e_{n-1} + d_n e_n + u_{n+1}e_{n+1}$ where $\{d_n\}$, $\{u_n\}$, $\{c_n\}$ are complex sequences whose absolute values are both bounded above and below by a positive constant. Suppose there exists a positive integer $k \geqslant 2$ and positive real numbers η, η' such that for some $\epsilon > 0$, the following conditions are satisfied for $0 \leqslant r \leqslant k-1$ and $n \in \mathbb{N}$:*

(i) $|\det T_k| \geqslant (1+\epsilon)|c_1||c_2|\cdots|c_k|, |\det T_k| \geqslant (1+\epsilon)|u_2|\cdots|u_{k+1}|,$

(ii)
$$\begin{aligned}
&|d_{kn-(k-1)+r}||\det A^{(r)}_{k-1,n}||\det B^{(r)}_{k-1,n}||c_{k(n-2)+2+r}|\cdots|c_{k(n-1)+r}|\\
&\quad\geqslant |\det B^{(r)}_{k-1,n}||\det A^{(r)}_{k-2,n}||u_{k(n-1)+r}||c_{k(n-2)+2+r}|\cdots|c_{k(n-1)+r}|\\
&\qquad+|\det B^{(r)}_{k-2,n}||\det A^{(r)}_{k-1,n}||c_{k(n-2)+2+r}|\cdots|c_{k(n-1)+r}| + |\det A^{(r)}_{k-1,n}||u^{(r)}_{k(n-2)+2+r}|\\
&\qquad+(1+\epsilon)|c_{k(n-2)+2+r}|\cdots|c_{k(n-1)+r}||c_{k(n-1)+1+r}|\cdots|c_{kn+r}||\det B^{(r)}_{k-1,n}|,
\end{aligned}$$

and

(iii)
$$\begin{aligned}
&|d_{kn-(k-1)+r}||\det A^{(r)}_{k-1,n}||\det B^{(r)}_{k-1,n}||u_{k(n-2)+3+r}|\cdots|u_{k(n-1)+1+r}|\\
&\quad\geqslant |\det B^{(r)}_{k-1,n}||\det A^{(r)}_{k-2,n}||c_{k(n-1)+1+r}||u_{k(n-2)+3+r}|\cdots|u_{k(n-1)+1+r}||u_{k(n-1)+2+r}|\\
&\qquad+|\det B^{(r)}_{k-2,n}||\det A^{(r)}_{k-1,n}||u_{k(n-2)+3+r}|\cdots|u_{k(n-1)+1+r}| + |\det A^{(r)}_{k-1,n}||c_{k(n-2)+1+r}|\\
&\qquad+(1+\epsilon)|\det B_{k-1}||u_{k(n-2)+3+r}|\cdots|u_{k(n-1)+1+r}||u_{k(n-1)+2+r}|\cdots|u_{kn+1+r}|,
\end{aligned}$$

(iv) $|\det A^{(r)}_{k-1,n}| > \eta, |\det B^{(r)}_{k-1,n}| > \eta'.$

Then $\{\|T_n^{-1}e_n\|\}$ and $\{\|T_n^{*^{-1}}e_n\|\}$ are bounded.

Here, we observe that if the given operator T is self adjoint, viz, when $u_j = \overline{c_{j-1}}$ the conditions (ii) *and* (iii) *are the same. Also there is only one inequality in* (i).

Proof. First we obtain a recurrence relation between $\det T_{kn}$, $\det T_{k(n-1)}$ and $\det T_{k(n-2)}$. For this purpose we consider the following $2k-1$ equations:

$$\begin{aligned}
(1):&\quad \det T_{kn} - d_{kn}\det T_{kn-1} + u_{kn}c_{kn-1}\det T_{kn-2} = 0.\\
(2):&\quad \det T_{kn-1} - d_{kn-1}\det T_{kn-2} + u_{kn-1}c_{kn-2}\det T_{kn-3} = 0.\\
\cdots&\\
(k-1):&\quad \det T_{kn-(k-2)} - d_{kn-(k-2)}\det T_{kn-(k-1)} + u_{kn-(k-2)}c_{kn-(k-1)}\\
&\quad \det T_{kn-k} = 0.\\
(k):&\quad \det T_{kn-(k-1)} - d_{kn-(k-1)}\det T_{kn-k} + u_{kn-(k-1)}c_{kn-k}\\
&\quad \det T_{kn-(k+1)} = 0.\\
(k+1):&\quad \det T_{kn-k} - d_{kn-k}\det T_{kn-(k+1)} + u_{kn-k}c_{kn-(k+1)}\\
&\quad \det T_{kn-(k+2)} = 0.\\
\cdots&\\
(2k-2):&\quad \det T_{kn-(2k-3)} - d_{kn-(2k-3)}\det T_{kn-(2k-2)} + u_{kn-(2k-3)}c_{kn-(2k-2)}\\
&\quad \det T_{kn-(2k-1)} = 0.\\
(2k-1):&\quad \det T_{kn-(2k-2)} - d_{kn-(2k-2)}\det T_{kn-(2k-1)} + u_{kn-(2k-2)}c_{kn-(2k-1)}\\
&\quad \det T_{kn-(2k)} = 0.
\end{aligned}$$

Multiply first $(k-1)$ equations by $\det A_j \det B_{k-1}$ for $j = 0, 1, 2, \ldots, k-2$ (with $\det A_0 = 1$) respectively. Multiply the last set of $(k-1)$ equations by $\det A_{k-1} \det B_j$ for $j = k-2, k-3, \ldots, 0$ (with $\det B_0 = 1$) respectively. Multiply the kth equation by $\det A_{k-1} \det B_{k-1}$ and add all the resulting equations. Then compute the coefficients of each $\det T_{kn-j}$. Calculating the coefficient of each $\det T_{kn-j}$ is easy because, the non-zero coefficients occur at the most only in three equations. With the straight forward calculations, one can see that all the coefficients except of $\det T_{kn}$, $\det T_{k(n-1)}$, $\det T_{k(n-2)}$ vanish. As a consequence, we obtain a recurrence relation in terms of these three determinants, namely,

$$\begin{aligned}
&\det B_{k-1,n}(\det T_{kn}) + [u_{kn-(k-2)}c_{kn-(k-1)} \det B_{k-1,n} \det A_{k-2,n} \\
&\quad -d_{kn-(k-1)} \det A_{k-1,n} \det B_{k-1,n} + \det B_{k-2,n} \det A_{k-1,n}](\det T_{k(n-1)}) \\
&\quad + \det A_{k-1,n} u_{kn-(2k-2)} c_{kn-(2k-1)} (\det T_{k(n-2)}) = 0.
\end{aligned}$$

In the same way, we can obtain a recurrence relation for $\det T_{kn+r}$, $\det T_{k(n-1)+r}$, $\det T_{k(n-2)+r}$, $1 \leqslant r \leqslant k-1$, where $A_{k-j,n}$, $B_{k-j,n}$ are replaced by $A^{(r)}_{k-j,n}$, $B^{(r)}_{k-j,n}$, and u_m, c_m, d_m are replaced by $u_{m+r}, c_{m+r}, d_{m+r}$ respectively. Next, we claim that for $n = 1, 2, 3, \ldots,$

$$|\det T_{kn}| \geqslant (1+\epsilon)|c_{k(n-1)+1}| \cdots |c_{kn}||\det T_{k(n-1)}|,$$

where $\det T_0 = 1$. The result is true for $n = 1$ by condition (i) of the hypothesis. Assume the result for $n-1$. Thus, we assume

$$|\det T_{k(n-1)}| \geqslant (1+\epsilon)|c_{k(n-2)+1}| \cdots |c_{k(n-1)}|.$$

Now, using the recurrence relation between $\det T_{kn}$, $\det T_{k(n-1)}$, $\det T_{k(n-2)}$, we get

$$\begin{aligned}
|\det B_{k-1,n}||\det T_{kn}| &\geqslant \big(|d_{kn-(k-1)}||\det A_{k-1,n}||\det B_{k-1,n}| - |u_{kn-(k-2)}| \\
&\quad \times |c_{kn-(k-1)}||\det B_{k-1,n}||\det A_{k-2,n}| - |\det B_{k-2,n}| \\
&\quad \times |\det A_{k-1,n}|\big)|\det T_{k(n-1)}| \\
&\quad -|\det A_{k-1,n}||u_{kn-(2k-2)}||c_{kn-(2k-1)}||\det T_{k(n-2)}| \\
&\geqslant (\cdots\cdots)|\det T_{k(n-1)}| \\
&\quad -\frac{1}{1+\epsilon}\frac{|\det A_{k-1,n}||u_{kn-(2k-2)}|}{|c_{k(n-2)+2}|\cdots|c_{k(n-1)}|}|\det T_{k(n-1)}| \\
&\quad \text{(by using induction hypothesis)} \\
&\geqslant (\cdots\cdots)|\det T_{k(n-1)}| \\
&\quad -\frac{|\det A_{k-1,n}||u_{kn-(2k-2)}|}{|c_{k(n-2)+2}|\cdots|c_{k(n-1)}|}|\det T_{k(n-1)}| \\
&= |d_{kn-(k-1)}||\det A_{k-1,n}||\det B_{k-1,n}| \\
&\quad \times |\det A_{k-1,n}||\det B_{k-1,n}||c_{k(n-2)+2}|\cdots|c_{k(n-1)}| \\
&\quad -|u_{kn-(k-2)}||\det B_{k-1,n}||\det A_{k-2,n}||c_{k(n-2)+2}|\cdots|c_{k(n-1)+1}| \\
&\quad -|\det B_{k-2,n}||\det A_{k-1,n}||c_{k(n-2)+2}|\cdots|c_{k(n-1)}| \\
&\quad -|\det A_{k-1,n}||u_{kn-(2k-2)}|\frac{1}{|c_{k(n-2)+2}|\cdots|c_{k(n-k)}|}|\det T_{k(n-1)}| \\
&\geqslant (1+\epsilon)|c_{k(n-1)+1}||c_{k(n-1)+2}|\cdots|c_{kn}||\det B_{k-1,n}||\det T_{k(n-1)}|
\end{aligned}$$

(by condition (ii) with $r = 0$). Thus, it follows that

$$|\det T_{kn}| \geqslant (1+\epsilon)|c_{k(n-1)+1}||c_{k(n-1)+2}|\cdots|c_{kn}||\det T_{k(n-1)}| \tag{9}$$

as $\det B_{k-1} \neq 0$. Similarly, we can show that for $r = 1, 2, \ldots, k-1$,

$$|\det T_{kn+r}| \geqslant (1+\epsilon)|c_{k(n-1)+1+r}||c_{k(n-1)+(2+r)}|\cdots|c_{kn+r}||\det T_{k(n-1)+r}|. \tag{10}$$

In a similar way, by using condition (iii), we can show that

$$|\det T_{kn+r}| \geqslant (1+\epsilon)|u_{k(n-1)+2+r}|\cdots|u_{kn+1+r}||\det T_{k(n-1)+r}| \tag{11}$$

for $r = 0, 1, 2, \ldots, k-1$. Next, we claim that

$$\left|\frac{\det T_{kn-1}}{\det T_{kn}}\right|, \left|\frac{\det T_{kn-2}}{\det T_{kn}}\right|, \ldots, \left|\frac{\det T_{kn-(k-1)}}{\det T_{kn}}\right|$$

are bounded.

In this case, we again need to consider $k-1$ equations, where each equation is the recurrence relation involving three determinants namely $\det T_{kn-i}$, $\det T_{kn-(i+1)}$, $\det T_{kn-(i+2)}$ for $i = 0, 1, 2, \ldots, k-2$. We may have to multiply each equation by appropriate coefficients (as it was done earlier) in such a way that the required bound, namely, $\left|\frac{\det T_{kn-j}}{\det T_{kn}}\right|$ is expressed in terms of $\left|\frac{\det T_{kn-k}}{\det T_{kn}}\right|$. Since by (9), $\left|\frac{\det T_{kn-k}}{\det T_{kn}}\right|$ is bounded by $\frac{1}{(1+\epsilon)\alpha^k}$ where $\alpha = \inf |c_i|$ we obtain the required bound for $\left|\frac{\det T_{kn-j}}{\det T_{kn}}\right|$. To avoid the computational complexity in a general k case, we illustrate the situation in the case $k = 3$. We have the following recurrence relations:

$$\det T_{3n} - d_{3n} \det T_{3n-1} + u_{3n}c_{3n-1} \det T_{3n-2} = 0, \tag{12}$$

$$\det T_{3n-1} - d_{3n-1} \det T_{3n-2} + u_{3n-1}c_{3n-2} \det T_{3n-3} = 0. \tag{13}$$

Multiply (13) by d_{3n} and adding with (12), we get

$$\det T_{3n} + (u_{3n}c_{3n-1} - d_{3n}d_{3n-1}) \det T_{3n-2} + d_{3n}u_{3n-1}c_{3n-2} \det T_{3n-3} = 0.$$

Thus,

$$\begin{aligned}\left|(d_{3n}d_{3n-1} - u_{3n}c_{3n-1})\frac{\det T_{3n-2}}{\det T_{3n}}\right| &= \left|(1 + d_{3n}u_{3n-1}c_{3n-2})\frac{\det T_{3n-3}}{\det T_{3n}}\right| \\ &\leqslant 1 + |d_{3n}||u_{3n-1}||c_{3n-2}|\frac{|\det T_{3n-3}|}{|\det T_{3n}|} \\ &\leqslant 1 + \frac{k_1k_2k_3}{(1+\epsilon)\alpha^k},\end{aligned}$$

where $k_1 = \sup_i |d_i|$, $k_2 = \sup_i |u_i|$, $k_3 = \sup_i |c_i|$. Thus

$$\left|\frac{\det T_{3n-2}}{\det T_{3n}}\right| \leqslant \frac{1}{|d_{3n}d_{3n-1} - u_{3n}c_{3n-1}|}\left(1 + \frac{k_1k_2k_3}{(1+\epsilon)\alpha^k}\right).$$

But Since $\{|\det A_{k-1}|\}$ is bounded below, we get the upper bound for $\frac{\det T_{3n-2}}{\det T_{3n}}$. Again, multiplying (12) by d_{3n-1} and (13) by $u_{3n}c_{3n-1}$, we get

$$d_{3n-1} \det T_{3n} + (u_{3n}c_{3n-1} - d_{3n-1}d_{3n}) \det T_{3n-1} + u_{3n}u_{3n-1}c_{3n-1}c_{3n-2} \det T_{3n-3} = 0.$$

Thus, we have

$$(d_{3n}d_{3n-1} - u_{3n}c_{3n-1})\frac{\det T_{3n-1}}{\det T_{3n}} = d_{3n-1} + u_{3n}u_{3n-1}c_{3n-1}c_{3n-2}\frac{\det T_{3n-2}}{\det T_{3n}}$$

from which it follows that

$$\left|\frac{\det T_{3n-1}}{\det T_{3n}}\right| \leqslant |d_{3n-1}| + k_2^2k_3^2\left|\frac{\det T_{3n-2}}{\det T_{3n}}\right|$$

which is bounded above.

Now consider

$$\begin{aligned}\|T_{kn}^{-1}e_{kn}\|^2 &= [c_1^2c_2^2\cdots c_{kn-1}^2 + c_2^2\cdots c_{kn-1}^2\det T_1^2 \\ &\quad + \cdots c_{kn-1}^2\det T_{kn-2}^2 + \det T_{kn-1}^2](\det T_n)^{-2} \\ &\leqslant \frac{1}{(\det T_{kn})^2}\left\{\left(\frac{1}{(1+\epsilon)^{2(n-1)}} + \frac{1}{(1+\epsilon)^{2(n-2)}} + \cdots + \frac{1}{(1+\epsilon)^2+1}\right)\right. \\ &\quad \left.\times(\det T_{kn-1}^2 + \det T_{kn-2}^2 + \cdots + \det T_{k(n-1)}^2)\right\}\end{aligned}$$

(by using (9) and (10)). Since $\frac{\det T_{kn-j}}{\det T_{kn}}$ is bounded for each j=1, 2, ..., k, it follows that $\{\|T_{kn}^{-1}e_{kn}\|\}$ is bounded. Similarly using condition (10), we can show that $\{\|T_{kn}^{*^{-1}}e_{kn}\|\}$ to be bounded. □

The conditions in Theorem 6.1 appear quite clumsy at first sight. However, for small values of k, these conditions take simple forms. We illustrate this for $k = 2$ and 3. These may be compared with the conditions given in Theorem 2.3, Remark 2.4 and Corollary 2.6 of [5].

Corollary 6.2. *Suppose* $Te_n = c_{n-1}e_{n-1} + d_ne_n + u_ne_{n+1}$ *where* $\{c_n\}, \{u_n\}, \{d_n\}$ *are sequences of numbers such that their absolute values are bounded above and below by a constant and satisfy the following conditions*:

There exists $\epsilon > 0$ *such that*

(i) $|d_i||d_{i-1}| \geqslant 2|c_{i-1}|(|u_i| + |c_{i-2}|)$ *and* $|d_i||d_{i-1}| \geqslant 2|u_i|(|c_{i-1}| + |u_{i-1}|)$,
(ii) $|d_i||d_{i+1}| \geqslant 2|u_{i+1}|(|c_i| + (1+\epsilon)|u_{i+2}|)$ *and* $|d_i||d_{i+1}| \geqslant 2|c_i|(|u_{i+1}| + (1+\epsilon)|c_{i+1}|)$ *for all* i,

then $\{\|T_n^{-1}e_n\|\}$ *and* $\{\|T_n^{*^{-1}}e_n\|\}$ *are bounded.*

Proof. We show that these conditions (i) and (ii) imply the conditions (i)–(iv) of Theorem 6.1 with $k = 2$. For this purpose, note that

$$\det B_{1,n} = \frac{d_{2n-2}}{c_{2n-2}u_{2n-1}}, \quad \det A_{1,n} = d_{2n}, \quad \det A_{0,n} = 1, \quad \text{and} \quad \det B_{0,n} = 1.$$

If we take $r = 0$ in condition (ii) of Theorem 6.1, after simplification it turns out to be

$$\begin{aligned}|d_{2n-1}||d_{2n}||d_{2n-2}| &\geqslant |d_{2n}||u_{2n-1}|(|c_{2n-2}| + |u_{2n-2}|) \\ &\quad + |d_{2n-2}||c_{2n-1}|(|u_{2n}| + (1+\epsilon)|c_{2n}|).\end{aligned}$$

Similarly condition (iii) leads to

$$\begin{aligned}|d_{2n}||d_{2n-1}||d_{2n-2}| &\geqslant |d_{2n}||c_{2n-2}|(|u_{2n-1}| + |c_{2n-3}|) \\ &\quad + |d_{2n-2}||u_{2n}|(|c_{2n-1}| + |u_{2n+1}|(1+\epsilon)).\end{aligned}$$

We obtain similar conditions for $r = 1$. Then the conditions (ii) and (iii) can be rewritten as

$$|d_{i-1}||d_i||d_{i+1}| \geqslant |d_{i+1}||u_i|(|c_{i-1}| + |u_{i-1}|) + |d_{i-1}||c_i|(|u_{i+1}| + (1+\epsilon)|c_{i+1}|)$$

and

$$|d_{i-1}||d_i||d_{i+1}| \geqslant |d_{i+1}||c_{i-1}|(|u_i| + |c_{i-2}|) + |d_{i-1}||u_{i+1}|(|c_i| + |u_{i+2}|(1+\epsilon)) \tag{14}$$

for all i. But we know that

$$Axy \geqslant Bx + Cy \text{ iff } (Ax - C)(Ay - B) \geqslant BC$$

whenever $A > 0$. These are satisfied if the following conditions hold:

$$\begin{aligned} |d_i||d_{i-1}| &\geqslant 2|c_{i-1}|(|u_i| + |c_{i-2}|),\\ |d_i||d_{i+1}| &\geqslant 2|u_{i+1}|(|c_i| + (1+\epsilon)|u_{i+2}|),\\ |d_i||d_{i-1}| &\geqslant 2|u_i|(|c_{i-1}| + |u_{i-1}|),\\ |d_i||d_{i+1}| &\geqslant 2|c_i|(|u_{i+1}| + (1+\epsilon)|c_{i+1}|). \end{aligned} \tag{15}$$

By simple calculations, one can verify that these conditions imply that $|\det T_2| \geqslant (1+\epsilon)|c_1||c_2|$ and

$$|\det T_2| \geqslant (1+\epsilon)|u_2||u_3|.$$

Thus condition (i) of Theorem 6.1 is satisfied. Also in this case, $\{|\det A_1^{(r)}|\}$ and $\{|\det B_1^{(r)}|\}$ are bounded below. Hence condition (iv) of Theorem 6.1 is satisfied, which completes the proof. $\square$

Next, we consider a special case of this corollary when $u_i = c_i = 1\ \forall i$.

Corollary 6.3. *Let $Te_n = e_{n-1} + d_n e_n + e_{n+1}$. If the sequence $\{d_i\}$ satisfy*

$$(|d_i||d_{i+1}| - 2)(|d_{i-1}||d_i| - 2) \geqslant 4 + \epsilon,$$

for each i, then $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$ are bounded.

Proof. The result follows from Eqs. (15) with $u_i = c_i = 1$. $\square$

Example 6.4. Let $u_i = c_i = 1$ for all i. Define $d_1 = d_2 = \frac{3}{2}$, $d_3 = 14$ and for $i = 2, 3, \ldots,$ $d_{3i-2} = \frac{3}{2} + \frac{1}{3i-2}$, $d_{3i-1} = \frac{9/4}{d_{3i-2}}$ (say), $d_{3i} = \frac{21}{d_{3i-1}}$.

First note that for each i, $d_{3i} \geqslant 14$. This can be proved by induction. Next, $d_{3i-1}d_{3i-2} = 9/4$, $d_{3i}d_{3i-1} = 21$ and $d_{3i+1}d_{3i} = d_{3i}\left(\frac{3}{2} + \frac{1}{3i+1}\right) \geqslant 14 \times 3/2 = 21$. Thus the conditions of Corollary 6.3 are satisfied with $\epsilon = 3/4$. Hence the corresponding operator T is invertible.

Corollary 6.5. *Let $Te_n = e_{n-1} + d_n e_n + e_{n+1}$. If the sequence $\{d_i\}$ satisfy*

$$\begin{aligned} |d_i|(|d_{i+1}||d_{i+2}| - 1)(|d_{i-1}||d_{i-2}| - 1) \geqslant{} & (|d_{i-1}||d_{i-2}| - 1)|d_{i+2}|\\ & + (|d_{i+1}||d_{i+2}| - 1)|d_{i-2}| + (|d_{i+1}||d_{i+2}| - 1)\\ & + (1+\epsilon)(|d_{i-1}||d_{i-2}| - 1) \end{aligned}$$

and $(|d_i||d_{i+1}| - 1) \geqslant \eta$ for all i where $\eta > 0$ with $|\det T_3| \geqslant (1+\epsilon)$, then $\{\|T_n^{-1} e_n\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$ are bounded.

Proof. The result follows from the theorem 6.1 with $k = 3$. $\square$

Example 6.6. Take $u_i = c_i = 1$ for all i. Define $d_1 = d_2 = \frac{3}{2}$, $d_3 = 9$ and for $i = 2, 3, \ldots,$ $d_{3i-2} = \frac{3}{2} + \frac{1}{3i-2}$, $d_{3i-1} = \frac{9/4}{d_{3i-2}} =$, $d_{3i} = \frac{9}{d_{3i-1}}$.

Note that for any $i \geqslant 2$,

$$(d_{3i-2}d_{3i-1} - 2)(d_{3i-1}d_{3i} - 2) = \left(\frac{9}{4} - 2\right)(9 - 2) = \frac{7}{4}.$$

Thus the conditions in Corollary 6.3 are not satisfied. On the other hand, for large values of i, $d_{3i-2} \simeq 3/2 \simeq d_{3i-1}$ and $d_{3i} = 6$. Thus it is easy to verify the conditions in Corollary 6.5 are satisfied.

Remark 6.7. Theorem 6.1 gives a set of sufficient conditions for each positive integer $k \geqslant 2$. If for any such k, the corresponding conditions are satisfied, then the theorem implies that the sequences $\{\|T_n^{-1}e_n\|\}$ and $\{\|T_n^{*^{-1}}e_n\|\}$ are bounded and hence T is invertible. In an earlier version of this paper, only the case $k = 2$ was discussed after proving Theorem 6.1. That version did not contain Corollary 6.5 and Example 6.6 which pertain to case $k = 3$. The referee's report on that version contained a following very important observation:

If the tridiagonal matrix T_n satisfies the assumptions in Corollary 6.3., then T_n can be factorized as $T_n = \Delta_n \widehat{T_n} \Delta_n$ where Δ_n is the diagonal matrix with jth diagonal entry $d_j^{1/2}$ and $\widehat{T_n}$ is the tridiagonal matrix whose jth row is given by

$$0, 0, \ldots, 0, (d_{j-1}d_j)^{-1/2}, 1, (d_j d_{j+1})^{-1/2}, 0, 0, \ldots, 0.$$

Noting that $\{\|\Delta_n^{-1}\|\}$ is bounded, it follows by applying the classical Gerschgorin theorem to $\hat{T}_n$ that the condition

$$|d_{j-1}d_j|^{-1/2} + |d_j d_{j+1}|^{-1/2} \leqslant 1 - \epsilon \tag{16}$$

for all j and some $\epsilon > 0$ is sufficient to imply the boundedness of $\{\|T_n^{-1}e_n\|\}$ and $\{\|T_n^{*^{-1}}e_n\|\}$. It turns out that the condition (16) is weaker than the condition

$$(|d_{j-1}d_j| - 2)(|d_j d_{j+1}| - 2) \geqslant 4 + \epsilon$$

for all j and some $\epsilon > 0$ given in Corollary 6.3.

In view of this comment, we were led to investigate the case $k = 3$ in Corollary 6.5 and Example 6.6. Note that for the operator in Example 6.6 taking $j = 3i - 1$, $d_{j-1}d_j = \frac{9}{4}$ and $d_j d_{j+1} = 9$. Thus

$$|d_{j-1}d_j|^{-1/2} + |d_j d_{j+1}|^{-1/2} = \frac{2}{3} + \frac{1}{3} = 1.$$

Hence the condition (16) is not satisfied. On the other hand, we have shown above that this operator satisfies assumptions in Corollary 6.5 and is hence invertible.

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