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AN ANALOGUE OF THE RIESZ-REPRESENTATION THEOREM

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Abstract. We prove the following analogue of the Riesz-Representation theorem for the space of quaternion-valued continuous functions on a compact Hausdorff space:

Let X be a compact Hausdorff space and $\psi : C(X, \mathbb{H}) \to \mathbb{H}$ be a bounded linear functional on a left quaternion normed linear space $C(X, \mathbb{H})$, then there exists a unique quaternion valued regular Borel measure λ on the σ-algebra of all Borel subsets of X such that

$$\psi(f) = \int_X f \, d\lambda, \ \textit{for all } f \in C(X, \mathbb{H})$$

and $\|\psi\| = |\lambda|(X)$, $|\lambda|$ is the total variation of λ.

Some basic results (needed in the proof of the main theorem) from the theory of quaternion measures are also proved. These include an analogue of Lusin's theorem and an analogue of the Radon-Nikodym theorem.

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1. Introduction

In this paper we give a characterization of bounded quaternion linear functionals on $C(X, \mathbb{H})$, the space of all quaternion-valued continuous functions on a compact Hausdorff space X. This can be considered as an analogue of the classical Riesz-Representation Theorem (Theorem 4.1).

Our proof of this main theorem requires certain basic results from two aspects of measure theory. Since these results had not been available in the literature, we had to prove these afresh. The first is the theory of integration of quaternion-valued functions with respect to a positive measure. This is developed in the next section. This section also contains an analogue of Lusin's theorem (Theorem 2.3) which deals with the problem of approximating quaternion-valued measurable functions by continuous functions. The third

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section deals with another aspect, namely the basic theory of quaternion-valued measures. We prove an analogue of the Radon-Nikodym theorem (Theorem 3.1) in this section. Section 4 contains a proof of the main theorem and the consequent characterization of the dual space of $C(X, \mathbb{H})$. For more results on quaternion normed linear spaces refer [1, 5, 7, 8, 9] and the references there in.

2. Integration of quaternion-valued function with respect to a positive measure

We denote the sets of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively, and that of quaternions by $\mathbb{H}$.

Let 1, i, j and k denote the usual quaternion members of a basis of $\mathbb{H}$. Thus any $q \in \mathbb{H}$ has a unique representation as $q = q_0 + q_1 i + q_2 j + q_3 k$ with q_0, q_1, q_2 and $q_3 \in \mathbb{R}$. Also, $q = z_1 + z_2 j$ where $z_1 = q_0 + q_1 i, z_2 = q_2 + q_3 i \in \mathbb{C}$.
For $q \in \mathbb{H}$, the conjugation is defined by $q^* = q_0 - q_1 i - q_2 j - q_3 k$. q_0 is called the real part of q, it is denoted by Re q and we have $q = \text{Re } q - i \text{ Re } iq - j \text{ Re } jq - k \text{ Re } kq$. Also for $p, q \in \mathbb{H}$, $\text{Re } pq = \text{Re } qp$.

$\mathbb{H}$ is a normed algebra with the norm defined by $|q| = (q^* q)^{1/2}$. Also, the norm satisfies, as in the complex case, $|pq| = |p||q|$ for all $p, q \in \mathbb{H}$. Elementary properties of quaternions can be found in any book on modern algebra (e.g. [2]).

We recall that a left (respectively right) quaternion normed linear space X is a real normed linear space which is also a left (respectively right) module on $\mathbb{H}$ and the norm satisfies $\|\alpha x\| = |\alpha| \, \|x\|$ (respectively $\|x\alpha\| = \|x\| \, |\alpha|$) for all $x \in X$, $\alpha \in \mathbb{H}$. A quaternion linear functional on a left (right) quaternion normed linear space X is a map $\phi : X \to \mathbb{H}$ satisfying

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) \quad (\text{respectively } \phi(x\alpha + y\beta) = \phi(x)\alpha + \phi(y)\beta) \quad \text{for all } x, y \in X, \alpha, \beta \in \mathbb{H}.$$

ϕ is said to be a real linear functional, if the above holds only for $\alpha, \beta \in \mathbb{R}$.

Similarly, the two-sided quaternion normed linear space and linear functionals on such spaces are defined. In this paper we will deal with the left quaternion normed linear spaces only. The results for the right and two-sided quaternion normed linear spaces can be obtained analogously. Torgasev in [7, 8] considered two-sided quaternion normed linear spaces.

Let $X_{\mathbb{R}}$ denote, X regarded as a real normed linear space. Note that the elements of X and $X_{\mathbb{R}}$ are the same.

The following Lemma gives a relationship between the quaternion linear functionals and real linear functionals on a quaternion normed linear space. The proof of this lemma is very similar to the method of obtaining linear functionals on a complex normed linear space in terms of real linear functionals on the underlying real normed linear space $X_{\mathbb{R}}$ (See [6]).

Lemma 2.1. *Let X be a (left) quaternion normed linear space. If ψ is a quaternion linear functional on X, then* $\mathrm{Re}\ \psi : X_{\mathbf{R}} \to \mathbf{R}$ *defined by*

$$(\mathrm{Re}\ \ \psi)(f) = \mathrm{Re}\ (\psi(f)),\ \ f \in X_{\mathbf{R}},$$

is a real linear functional on $X_{\mathbb{R}}$ *and* $\|\psi\| = \|\mathrm{Re}\ \ \psi\|$*. Moreover, for all* $f \in X$,

$$\psi(f) = \mathrm{Re}\ \ \psi(f) - i\mathrm{Re}\ \ \psi(if) - j\mathrm{Re}\ \ \psi(jf) - k\mathrm{Re}\ \ \psi(kf)$$

Conversely, if ϕ is a real linear functional on $X_{\mathbf{R}}$ *and let*

$$\psi(f) = \phi(f) - i\phi(if) - j\phi(jf) - k\phi(kf) \text{ for all } f \in X,$$

then ψ is a quaternion linear functional on X, $\mathrm{Re}\ \psi = \phi$ *and* $\|\psi\| = \|\phi\|$.

We need few facts from the theory of integration of quaternion-valued functions with respect to a positive measure. This theory is almost similar to the theory of integration of complex valued measurable functions [3, 4]. Let $(X, \mathbb{M})$ be a measurable space and μ a positive measure on $(X, \mathbb{M})$. A quaternion-valued measurable function f on $(X, \mathbb{M})$ is said to be *integrable* if $\int_X |f| d\mu < \infty$. It is easy to check that $f = f_0 + if_1 + jf_2 + kf_3 = g + hj$ is integrable if and only if the real functions f_0, f_1, f_2 and f_3 are integrable or, equivalently, if and only if the complex functions $g = f_0 + if_1$ and $h = f_2 + if_3$ are integrable. For an integrable f we define

$$\int_X f d\mu = \int_X f_0 d\mu + i \int_X f_1 d\mu + j \int_X f_2 d\mu + k \int_X f_3 d\mu.$$

Thus, for an integrable f, $\int_X f\ d\mu \in \mathbb{H}$.

It can be easily shown that the integral defined in this way has the following properties:

Let f, g be quaternion-valued integrable functions on X and $p, q \in \mathbb{H}$. Then

1. $\int_X (f+g) d\mu = \int_X f\ d\mu + \int_X g\ d\mu$
2. $\int_X p\ fq\ d\mu = p\left(\int_X f\ d\mu\right) q$
3. $\left|\int_X f\ d\mu\right| \leq \int_X |f|\ d\mu$
4. $\left(\int_X f\ d\mu\right)^* = \int_X f^* d\mu$

For $1 \leq p < \infty$, we define $L^p_{\mathbb{H}}(\mu)$ as the space of all equivalence classes of quaternion-valued measurable functions f on X for which $|f|^p$ is integrable. It is easy to prove that if $f = g + hj$, where g and h are complex functions, then $f \in L^p_{\mathbb{H}}(\mu)$ if and only if $g, h \in L^p_{\mathbb{C}}(\mu)$, the complex $L^p(\mu)$. $L^p_{\mathbb{H}}(\mu)$ is left (also right, two-sidedd) quaternion normed linear space with the $p-norm$ $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$. $L^\infty_{\mathbb{H}}(\mu)$ denotes the space of all equivalence classes of essentially bounded quaternion-valued measurable functions on a measure space with the essential supremum norm.
The proof of the following results follow on the similar lines of their complex versions ([4] p.24).

Theorem 2.2. *Let μ be a positive measure on X.*

a) (Lebesgue's Dominated Convergence Theorem) Suppose $\{f_n\}$ is a sequence of quaternion-valued measurable functions on X such that

$$f(x) = \lim_{n \to \infty} f_n(x)$$

exists for every $x \in X$. If there is an integrable function g such that

$$|f_n(x)| \leq g(x) \ (n = 1, 2, 3, ...; x \in X),$$

then $f \in L^1_{\mathbb{H}}(\mu)$, and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

b) Suppose $f \in L^1_{\mathbb{H}}(\mu)$, and $\int_E f \, d\mu = 0$ for all $E \in \mathbb{M}$. Then $f = 0$ a.e. on X.

c) Suppose $\mu(X) < \infty$, $f \in L^1_{\mathbb{H}}(\mu)$, S is a closed set in $\mathbb{H}$ and the averages $\frac{1}{\mu(E)} \int_E f \, d\mu$ lie in S for every $E \in \mathbb{M}$, with $\mu(E) > 0$, then $f(x) \in S$ for almost all x.

Let X be a compact Hausdorff space. We denote the set of all $\mathbb{F} - Valued$ continuous functions on X, by $C(X, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $C(X, \mathbb{F})$ is a normed linear space over $\mathbb{F}$, with the supremum norm. $C(X, \mathbb{H})$ is a left (also right, two-sidedd) quaternion normed linear space with the supremum norm.

We prove the following analogue of Lusin's theorem which can be used to approximate a quaternion-valued measurable function by a continuous quaternion-valued function. This will be used in proving an analogue of the Riesz-Representation Theorem in Section 4.

Theorem 2.3. *[(Analogue of Lusin's theorem)] Suppose f is a quaternion-valued measurable function on a compact Hausdorff space X, μ is a positive regular Borel measure on X, $\mu(X) < \infty$ and $\epsilon > 0$. Then there exists a $g \in C(X, \mathbb{H})$ such that*

$$\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$$

Further, it is possible to choose g in such a way that

$$\sup\{|g(x)| : x \in X\} \leq ess. \sup |f|.$$

Proof. f can be written as $f = f_1 + f_2 j$, for some complex measurable functions f_1 and f_2. As $\mu(X) < \infty$, applying Lusin's Theorem ([4], p. 55) to the complex functions f_1 and f_2 there exist $h_1, h_2 \in C(X, \mathbb{C})$ such that the sets

$$A_1 = \{x \in X : f_1(x) \neq h_1(x)\},$$
$$A_2 = \{x \in X : f_2(x) \neq h_2(x)\}$$

satisfy $\mu(A_1) < \epsilon/2$ and $\mu(A_2) < \epsilon/2$. Let $A = A_1 \bigcup A_2$ then $\mu(A) < \epsilon$ and for all $x \in X \setminus A$, $f_1(x) = h_1(x)$ and $f_2(x) = h_2(x)$.
Let $h = h_1 + h_2 j$ then $h \in C(X, \mathbb{H})$, $f = h$ on $X \setminus A$ and

$$\{x \in X : f(x) \neq h(x)\} \subseteq A.$$

This proves the first part.

Now, let

$$r = ess. \sup |f|,$$

and define $\varphi(q) = q$ if $|q| \leq r$ and $\varphi(q) = rq/|q|$ if $|q| > r$. Then φ is a continuous mapping of $\mathbb{H}$ onto the closed disc in $\mathbb{H}$ of radius r. Let $g = \varphi \circ h$. Then $\sup\{|g(x)| : x \in X\} \leq r = ess. \sup |f|$ and

$$\{x \in X : g(x) \neq f(x)\} \subseteq \{x \in X : h(x) \neq f(x)\} \cup \{x \in X : |f(x)| > r\}$$

thus

$$\mu(\{x \in X : g(x) \neq f(x)\} < \epsilon \text{ as } \mu\{x \in X : |f(x)| > r\} = 0 \qquad \square$$

Corollary 2.4. *Let X and μ be as in Theorem 2.3 and f be a quaternion-valued measurable function on X such that ess. $\sup |f| \leq 1$. Then there exist $\{g_n\} \in C(X, \mathbb{H})$ such that $||g_n|| \leq 1$, $\lim_{n\to\infty} g_n(x) = f(x)$ for almost all x; that is $g_n \to f$ almost everywhere.*

Proof. By the Theorem 2.3 to each n, there exists $g_n \in C(X, \mathbb{H})$ such that

$$||g_n|| \leq ess. \sup |f| \leq 1$$

and $\mu(E_n) < 2^{-n}$, where E_n is the set of all x at whcih $f(x) \neq g_n(x)$. Now, since

$$\sum_{n=1}^{\infty} \mu(E_n) \leq 1 \ < \ \infty,$$

almost every x lie only in finitely many of the sets E_n ([4] p.31). Thus $f(x) = g_n(x)$ for all large enough n, for almost all x. That is limit $g_n(x) = f(x)$ for almost all x. Thus $g_n \to f$ almost everywhere. □

3. Quaternion measures

Let $\mathbb{M}$ be a $\sigma - algebra$ in a set X. Call a countable collection $\{E_i\}$ of members of $\mathbb{M}$ a partition of E if $E_i \bigcap E_j = \phi$ whenever $i \neq j$, and $E = \bigcup E_i$. A *quaternion measure* λ on $\mathbb{M}$ is a quaternion-valued function on $\mathbb{M}$ such that

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$$

for every partition $\{E_i\}$ of E.
If λ is a quaternion measure on $\mathbb{M}$, then λ can be written as $\lambda = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$, $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are real-valued set functions on $\mathbb{M}$. It is easy to see that λ is a quaternion measure on $\mathbb{M}$, if and only if $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are real measures on $\mathbb{M}$. Also, if we write $\nu_1 = \lambda_0 + \lambda_1 i$, $\nu_2 = \lambda_2 + \lambda_3 i$ then $\lambda = \nu_1 + \nu_2 j$, and it is easy to see that λ is a quaternion measure if and only if ν_1 and ν_2 are complex measures on $\mathbb{M}$.

Define a set function $\mid \lambda \mid$ on $\mathbb{M}$ by

$$|\lambda|(E) = sup \sum_{i=1}^{\infty} |\lambda(E_i)|$$

where the supremum is taken over all partitions $\{E_i\}$ of E.
We call the set function $|\lambda|$ the *total variation* of λ. Like complex measures, it can be proved that the total variation $|\lambda|$ of a quaternion measure λ is a positive measure on $\mathbb{M}$. Also, since the total varition of a complex measure is a finite measure, for every partition $\{E_i\}$ of E, we have

$$\sum_{i=1}^{\infty} |\lambda(E_i)| = \sum_{i=1}^{\infty} |\nu_1(E_i) + \nu_2(E_i)j| \leq \sum_{i=1}^{\infty} |\nu_1(E_i)| + |\nu_2(E_i)| \leq |\nu_1|(E) + |\nu_2|(E).$$

Now by taking the supremum of the left hand side over all partitions of E, we have

$$|\lambda|(E) \ \leq \ |\nu_1|(E) + |\nu_2|(E) < \infty, \ \textit{for all} \ E \in \mathbb{M}.$$

Thus, the total varition $|\lambda|$ of a quaternion measure λ is a finite positive measure.

A quaternion measure λ defined on the $\sigma - algebra$ of all Borel subsets of a topological space X is said to be *regular* if and only if the total variation $|\lambda|$ is regular.

Let μ be a positive measure on a σ-algebra $\mathbb{M}$, and let λ be an arbitrary measure on $\mathbb{M}$; λ may be positive or a quaternion measure. We say that λ is *absolutely continuous* with respect to μ, and write $\lambda \ll \mu$ if $\lambda(E) = 0$ for every $E \in \mathbb{M}$ for which $\mu(E) = 0$.
The following theorem is an analogue of the Radon-Nikodym Theorem [4].

Theorem 3.1. *(Analogue of the Radon-Nikodym Theorem) Let μ be a positive $\sigma - finite$ measure and λ a quaternion measure on $\mathbb{M}$ such that $\lambda \ll \mu$, then there exists a unique $h \in L^1_{\mathbb{H}}(\mu)$ such that*

$$\lambda(E) = \int_E h \, d\mu, \quad for \ all \ E \in \mathbb{M}. \tag{1}$$

Proof. λ can be written as $\lambda = \lambda_1 + \lambda_2 j$, where λ_1 and λ_2 are complex measures. Since $\lambda \ll \mu$, whenever $\mu(E) = 0$, we have $\lambda(E) = 0$, which implies that $\lambda_1(E) = 0$, and $\lambda_2(E) = 0$, hence $\lambda_1 \ll \mu$, and $\lambda_2 \ll \mu$. By the Radon-Nikodym theorem for complex measures, there exist h_1 and $h_2 \in L^1_{\mathbb{C}}(\mu)$ such that $\lambda_1(E) = \int_E h_1 \, d\mu$ and $\lambda_2(E) = \int_E h_2 \, d\mu$ for all $E \in \mathbb{M}$. If $h = h_1 + h_2 j$ then

$$\int_X |h| \, d\mu \;\leq\; \int_X (|h_1| + |h_2|) d\mu \;<\; \infty$$

as h_1 and $h_2 \in L^1_{\mathbb{C}}(\mu)$. Thus $h \in L^1_{\mathbb{H}}(\mu)$ and

$$\lambda(E) = \lambda_1(E) + \lambda_2(E) j = \int_E h \, d\mu.$$

Further, if $h, g \in L^1_{\mathbb{H}}(\mu)$ satisfy equation (3.1) then

$$\int_E (h - g) \, d\mu = 0, \quad for \ all \ E \in \mathbb{M},$$

hence by Theorem 2.2(b), $h = g$ almost everywhere. This proves the uniqueness of h. □

Corollary 3.2. *Let λ be a quaternion measure on a σ-algebra $\mathbb{M}$ in X. Then, there is a quaternion-valued measurable function h on X such that $|h(x)| = 1$ for all $x \in X$, and such that $\lambda(E) = \int_E hd|\lambda|$, for all $E \in \mathbb{M}$. In other words $d\lambda = hd|\lambda|$.*

Proof. Clearly $\lambda \ll |\lambda|$. Since $|\lambda|$ is finite, by Theorem 3.1, there exists $h \in L^1_{\mathbb{H}}(|\lambda|)$ such that

$$\lambda(E) = \int_E h \, d|\lambda|, \quad E \in \mathbb{M}. \tag{2}$$

To prove $|h(x)| = 1$ for all $x \in X$. Let $A_r = \{x : |h(x)| < r\}$ where r is some positive number, and let $\{E_j\}$ be a partition of A_r. Then

$$\sum_j |\lambda(E_j)| \leq \sum_j \int_{E_j} |h|\, d|\lambda| \leq r \sum_j |\lambda|(E_j) = r|\lambda|(A_r).$$

Taking the supremum of the left side over all partitions of A_r we get $|\lambda|(A_r) \leq r|\lambda|(A_r)$.
If $r < 1$ the above inquality implies that $|\lambda|(A_r) = 0$. Thus $|h| \geq 1$ almost everywhere.
On the other hand, if $E \in \mathbb{M}$ is such that $|\lambda|(E) > 0$, then from equation (2) we have $\left|\frac{1}{|\lambda|(E)} \int_E h\, d|\lambda|\right| = \frac{|\lambda(E)|}{|\lambda|(E)} \leq 1$. Thus the averages $\frac{1}{|\lambda|(E)} \int_E h\, d|\lambda|$ lie in the closed unit disc of $\mathbb{H}$. By Theorem of averages (Theorem 2.2(c)) the range of h is contained in the closed unit disc of $\mathbb{H}$ almost everywhere, that is $|h| \leq 1$ almost everywhere. Thus $|h| = 1$ almost everywhere.
Let $B = \{x \in X : |h(x)| \neq 1\}$. Then $|\lambda|(B) = 0$. If we redefine h on B by $h(x) = 1$, $x \in B$ we obtain a function h with the desired properties. □

4. Analogue of the Riesz-Representation theorem

Let X be a compact Hausdorff space. In this section we give a characterization of the bounded quaternion linear functionals on the left quaternion normed linear space $C(X, \mathbb{H})$.

Let λ be a quaternion-valued Borel measure on X. By Corollary 3.2, there is a quaternion-valued measurable function h on X such that $|h(x)| = 1$ for all $x \in X$ and

$$\lambda(E) = \int_E h d|\lambda|, \qquad \text{for every Borel set } E.$$

By comparing the coefficients, we have

$$\lambda_n(E) = \int_E h_n d|\lambda|, \quad n = 0, 1, 2, 3,$$

where $\lambda = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$ and $h = h_0 + h_1 i + h_2 j + h_3 k$.
Therefore $d\lambda_n = h_n d|\lambda|$, $n = 0, 1, 2, 3$.

We now define an i ral of a quaternion-valued bounded measurable function f with respect to aternion measure λ by

$$\int_E f d\lambda := \int_E f h d|\lambda|.$$

Equivalently

$$\int_E f d\lambda = \int_E f h d|\lambda| = \int_E f h_0 d|\lambda| + \int_E f h_1 d|\lambda| \cdot i + \int_E f h_2 d|\lambda| \cdot j + \int_E f h_3 d|\lambda| \cdot k$$

$$= \int_E f d\lambda_0 \quad \int_E f d\lambda_1 \cdot i + \int_E f d\lambda_2 \cdot j + \int_E f d\lambda_3 \cdot k.$$

If λ is a quaternion-valued regular Borel measure on X then it is easy to see that, the map $\psi_\lambda : C(X,\mathbb{H}) \to \mathbb{H}$ defined by

$$\psi_\lambda(f) = \int_X f d\lambda = \int_X fh\ d|\lambda|, \quad f \in C(X,\mathbb{H})$$

is a quaternion linear functional on the left quaternion normed linear space $C(X,\mathbb{H})$. Further, ψ_λ is bounded since, $|\psi_\lambda(f)| = \left|\int_X fh\ d|\lambda|\right| \leq \|f\|\ |\lambda|(X)$. Here $h \in L^1_{\mathbb{H}}(|\lambda|)$ is as in Corollary 3.2.

If $C(X,\mathbb{H})$ is regarded as a right quaternion normed linear space then $\psi_\lambda : C(X,\mathbb{H}) \to \mathbb{H}$ defined by $\psi_\lambda(f) = \int_X d\lambda\ f$, for all $f \in C(X,\mathbb{H})$ is a bounded (right) quaternion linear functional on the right quaternion normed linear space $C(X,\mathbb{H})$.

The following analogue of the Riesz-Representation Theorem shows that all the bounded quaternion linear functionals on $C(X,\mathbb{H})$ are of this form.

Theorem 4.1. *(Analogue of Riesz-Representation Theorem) Let X be a compact Hausdorff space and $\psi : C(X,\mathbb{H}) \to \mathbb{H}$ be a bounded linear functional on the (left) quaternion normed linear space $C(X,\mathbb{H})$. Then there exists a unique quaternion-valued regular Borel measure λ on X such that*

$$\psi(f) = \int_X f\ d\lambda, \quad \text{for all } f \in C(X,\mathbb{H}) \tag{3}$$

and $\|\psi\| = |\lambda|(X)$

Proof. First we shall prove the existence of λ.

Regard $C(X,\mathbb{H})$ as a real vector space and let $\phi : C(X,\mathbb{H}) \to \mathbb{R}$ be defined by

$$\phi(f) = \text{Re}\ \ \psi(f), \quad f \in C(X,\mathbb{H})$$

By Lemma 2.1, ϕ is a bounded real linear functional. Define $\phi_0 : C(X,\mathbb{R}) \to \mathbb{R}$ by $\phi_0(f) = \phi(f)$, $f \in C(X,\mathbb{R})$. Then ϕ_0 is a bounded real linear functional on $C(X,\mathbb{R})$, $\|\phi_0\| \leq \|\phi\|$. By the Riesz-Representation theorem for the space $C(X,\mathbb{R})$ [4], there exists a real regular Borel measure λ_0 on the σ-algebra of all Borel subsets of X such that

$$\phi_0(f) = \int_X f\ d\lambda_0 \quad \text{for all } f \in C(X,\mathbb{R}).$$

Similarly, $\phi_1(f) = \phi(if)$, $\phi_2(f) = \phi(jf)$, $\phi_3(f) = \phi(kf)$, $f \in C(X,\mathbb{R})$, are bounded real linear functionals on $C(X,\mathbb{R})$ and hence, as above, there exist a real regular Borel measures λ_1, λ_2 and λ_3 on X such that

$$\phi_1(f) = \int_X f\ d\lambda_1, \phi_2(f) = \int_X f\ d\lambda_2, \phi_3(f) = \int_X f\ d\lambda_3, \ \text{for all } f \in C(X,\mathbb{R}).$$

Now, let $f = f_0 + if_1 + jf_2 + kf_3 \in C(X, \mathbb{H})$. Then $f_0, f_1, f_2, f_3 \in C(X, \mathbb{R})$ and

$$\begin{aligned} \text{Re } \psi(f) &= \phi(f_0 + if_1 + jf_2 + kf_3) = \phi(f_0) + \phi(if_1) + \phi(jf_2) + \phi(kf_3) \\ &= \int_X f_0 \, d\lambda_0 + \int_X f_1 \, d\lambda_1 + \int_X f_2 \, d\lambda_2 + \int_X f_3 \, d\lambda_3. \end{aligned}$$

Now, by Lemma 2.1,

$$\begin{aligned} \psi(f) &= \phi(f) \ - i\ \phi(if) \ - j\ \phi(jf) \ - k\ \phi(kf) \\ &= \phi(f_0 + if_1 + jf_2 + kf_3) - i\ \phi(if_0 - f_1 + kf_2 - jf_3) \\ &\quad - j\ \phi(jf_0 - kf_1 - f_2 + if_3) - k\ \phi(kf_0 + jf_1 - if_2 - f_3) \\ &= \int_X f_0 \, d\lambda_0 + \int_X f_1 \, d\lambda_1 + \int_X f_2 \, d\lambda_2 + \int_X f_3 \, d\lambda_3 \\ &\quad - i[-\int_X f_1 \, d\lambda_0 + \int_X f_0 \, d\lambda_1 - \int_X f_3 \, d\lambda_2 + \int_X f_2 \, d\lambda_3] \\ &\quad - j[-\int_X f_2 \, d\lambda_0 + \int_X f_3 \, d\lambda_1 + \int_X f_0 \, d\lambda_2 - \int_X f_1 \, d\lambda_3] \\ &\quad - k[-\int_X f_3 \, d\lambda_0 - \int_X f_2 \, d\lambda_1 + \int_X f_1 \, d\lambda_2 + \int_X f_0 \, d\lambda_3] \\ &= \int_X (f_0 + if_1 + jf_2 + kf_3) d\lambda_0 + \int_X (f_1 - if_0 - jf_3 + kf_2) d\lambda_1 \\ &\quad + \int_X (f_2 + if_3 - jf_0 - kf_1) d\lambda_2 + \int_X (f_3 - if_2 + jf_1 - kf_0) d\lambda_3 \\ &= \int_X f \, d\lambda_0 + \int_X f \cdot \ -i \ d\lambda_1 + \int_X f \ \cdot -j \ d\lambda_2 + \int_X f \ \cdot -k \ d\lambda_3 \end{aligned}$$

$= \displaystyle\int_X f \, d\lambda$ where $\lambda = \lambda_0 - i\lambda_1 - j\lambda_2 - k\lambda_3$ is a quaternion-valued Borel measure on X. Also, since $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are regular Borel measures, λ is regular. This proves the existence of λ. Now, by Corollary 3.2 there exists $h \in L^1_{\mathbb{H}}(|\lambda|)$ such that $|h(x)| = 1$ for all $x \in X$ and $d\lambda = hd|\lambda|$. Thus, for all $f \in C(X, \mathbb{H})$

$$|\psi(f)| = |\int_X f \, d\lambda| = |\int_X fh \, d|\lambda|| \leq \int_X |f| \, d|\lambda| \leq ||f|| \ |\lambda|(X).$$

Thus $||\psi|| \leq |\lambda|(X)$. Further, since $|\lambda|(X) < \infty$, by Corollary 2.4 there exists $\{g_n\} \in C(X, \mathbb{H})$ such that $g_n \to \bar{h}$ almost everywhere and

$$||g_n|| \leq ess. \sup |\bar{h}| = 1, \quad n = 1, 2, 3....$$

Now, $g_n h \to \bar{h}h = 1$ almost everywhere and $|\lambda|(X) < \infty$, therefore, by the dominated convergence theorem (Theorem 2.2 (a))

$$\psi(g_n) = \int_X g_n \, d\lambda = \int_X g_n h \, d|\lambda| \to \int_X d|\lambda| = |\lambda|(X).$$

Therefore $||\psi|| = |\lambda|(X)$.
Now to prove the Uniqueness of the measure λ, suppose λ_1 and λ_2 are quaternion-valued regular Borel measures satisfying (3). Then $\lambda = \lambda_1 - \lambda_2$ is a quaternion-valued regular Borel measure which satisfies,

$$\int_X f\, d\lambda = 0, \text{ for all } f \in C(X, \mathbb{H}).$$

Now by Corollary 3.2 there exists $h \in L^1_{\mathbb{H}}(|\lambda|)$ such that $|h(x)| = 1$ for all $x \in X$ and $d\lambda = hd|\lambda|$. Let $\epsilon > 0$. By Theorem 2.3, there exists $g \in C(X, \mathbb{H})$ such that $||g|| \leq sup\{|\bar{h}(x)| : x \in X\} = 1$ and $|\lambda|(E) < \epsilon$ where $E := \{x \in X : g(x) \neq \bar{h}(x)\}$. Since $0 = \int_X g\, d\lambda = \int_X gh\, d|\lambda|$, we have $|\lambda|(X) = \int_X (\bar{h} - g)hd|\lambda| \leq \int_X |\bar{h} - g|\, d|\lambda| = \int_E |\bar{h} - g|\, d|\lambda| \leq 2|\lambda|(E) < 2\epsilon$. Since ϵ was arbitrary, $|\lambda|(X) = 0$. Thus $\lambda = 0$, hence $\lambda_1 = \lambda_2$. □

Remark 4.2 Let X be a compact Hausdorff space and let $M(X, \mathbb{H})$ denote the set of all quaternion-valued regular Borel measures λ on X. Then, $M(X, \mathbb{H})$ is a left (also right, two-sided) quaternion vector space. For $\lambda \in M(X, \mathbb{H})$, define $||\lambda|| = |\lambda|(X)$. This defines a norm on $M(X, \mathbb{H})$ making it a Banach space. Let ψ be a bounded quaternion linear functional on the left quaternion normed linear space $C(X, \mathbb{H})$ and $F(\psi) = \lambda$, where $\lambda \in M(X, \mathbb{H})$ is as given by Theorem 4.1.

Note that $F(\psi_1 + \psi_2) = F(\psi_1) + F(\psi_2)$. However, in general (due to the non-commutativity of quaternions) for $q \in \mathbb{H}$, $q\psi$ is not a quaternion linear functional on the left quaternion normed linear space $C(X, \mathbb{H})$. On the other hand, ψq defined by $\psi q(f) = \psi(f)q$, $q \in \mathbb{H}$, is a quaternion linear functional on $C(X, \mathbb{H})$, and we have $F(\psi q) = F(\psi)q$. Thus, the dual space of the left quaternion normed linear space $C(X, \mathbb{H})$ is a right module over $\mathbb{H}$ and by Theorem 4.1, F is a quaternion linear isometry of the dual space of $C(X, \mathbb{H})$ onto $M(X, \mathbb{H})$, when both are regarded as right quaternion Banach spaces, and hence right quaternion Banach space $M(X, \mathbb{H})$ can be identified with the dual space of the left quaternion normed linear space $C(X, \mathbb{H})$.

Similarly, the analogue of Theorem 4.1 for the right quaternion normed linear space $C(X, \mathbb{H})$ and the characterization of the dual space of the right quaternion normed linear space $C(X, \mathbb{H})$ with the left quaternion normed linear space $M(X, \mathbb{H})$ can be obtained.

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References

[1] Kulkarni, S.H., Representation of a class of real B^*-algebra as algebra of quaternion-valued functions, Proc. Amer. Math. Soc. 116 (1992), 61-66.

[2] Luthar, I.S., Passi, I.B.S., Algebra, Narosa publishing House, Vol. 2, 1999 (India).

[3] Natarajan, S., Viswanath, K., Quaternionic Representations of Compact Metric Groups, J. Math. Phys. 8 (1967), 582-589.

[4] Rudin, W., Real and Complex Analysis, McGraw-Hill International Editions, 3rd Edition, 1986.

[5] Sharma, C.S., Almeida, D.F., Additive functionals and operators on a quaternionic Hilbert space, J. Math. Phys. 30(2) (1989), 369-375.

[6] Suhomlinov, G., On extensions of linear functions in complex and quaternionic linear spaces, Mat. Sbornik, 3 (1938), 353-358.

[7] Torgasev, A., On the symmetric quaternionic Banach algebras, I (Gelfand Theory), Nouvelle serie, tome 24 (38), (1978), 173-188.

[8] Torgasev, A., On reflexivity of a quaternion normed space, Novi Sad J. Math. 27 (1997), no. 2, 57-64.

[9] Zhang, F., Quaternions and matrices of quaternions, Linear Algebra and its Applications 57 (1997), 21-57.

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