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THE GROUP OF INVERTIBLE ELEMENTS OF A REAL BANACH ALGEBRA

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Abstract. The following result is proved: Let A be a commutative real Banach algebra with unit 1. Let G denote the group of invertible elements of A and let G_1 be the connected component of G containing 1. If the quotient group G/G_1 contains an element of finite order other than G_1, then the order of such an element must be 2. If the group G/G_1 is of finite order, then its order must be 2^n for some nonnegative integer n.

1. Introduction

Let A be a commutative complex Banach algebra with unit 1. Let G denote the group of invertible elements of A and let G_1 be the connected component of G containing 1. Then G_1 is also a group and $G_1 = \{\exp(a) : a \in A\}$. (See [1], [7].) A well known theorem due to Lorch says that the quotient group G/G_1 contains no element of finite order except the unit element G_1. (See [7], Theorem 10.44.) This is false for real Banach algebras as the trivial example $A = \mathbb{R}$ shows. Here G is the set of all nonzero real numbers, G_1 is the set of all positive real numbers and the element $-1.G_1$ is of order two in G/G_1. In fact G/G_1 has only two elements. A less trivial example is obtained by considering $A = C_{\mathbb{R}}([0,1])$, the real Banach algebra of all real valued continuous functions defined on the interval $[0,1]$ with pointwise operations and the supremum norm. In this case also, the group G/G_1 is of order two. (See [7], Exercise 21, Chapter 10.)

This raises a natural question: What are the possible orders of elements in the group G/G_1 of a real Banach algebra A? This question has a very interesting answer, namely, 2 apart from of course one and infinity. This also means that

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if G/G_1 is a finite group, then its order must be 2^n for some natural number n. The aim of this note is to present a proof of this result. A careful reader may observe that the main ideas of the proof are already present in the known proof of the theorem for complex Banach algebras. Though it seems natural that such a result should have been expected, it does not seem to have appeared in print.

Since every complex algebra is also a real algebra, the class of real Banach algebras is a larger class. On the other hand, the class of complex Banach algebras is a very well studied class, with rich theory mainly due to the possibility of using very rich complex function theory. There have been attempts in the literature to extend the results of complex Banach algebras to the larger class of real Banach algebras. An account of such attempts can be found in the monographs [3] and [5]. Many of these attempts consist in considering some theorem about complex Banach algebras and showing that a similar theorem also holds for real Banach algebras. In contrast, the present note considers a theorem which is true for complex Banach algebras but not true for real Banach algebras. Another instance of a similar theorem is the well known Gleason-Kahane-Zelazko Theorem. (See [2])

2. Main result

We refer to [3] for the basic theory of real Banach algebras. Here we recall a few concepts needed to prove our main result. Let A be a commutative real Banach algebra with unit 1. Let G denote the group of invertible elements of A and let G_1 be the connected component of G containing 1. For $s \in \mathbb{R}$, we shall identify the element $s1$ with s. For $a \in A$, the *spectrum* $\sigma(a)$ of a is defined by

$$\sigma(a) := \{s + it \in \mathbb{C} : (s - a)^2 + t^2 \notin G\}$$

It is clear that $s + it \in \sigma(a)$ if and only if $s - it \in \sigma(a)$. Also if s is real, then $s \in \sigma(a)$ if and only if $s - a \notin G$. We also need the following: If m is a natural number, then $\sigma(a^m) = \{\lambda^m : \lambda \in \sigma(a)\}$. This is a special case of the Spectral Mapping Theorem for real Banach algebras. (See [3], [4] for a proof.) A consequence of the Spectral Mapping Theorem is that if the spectrum $\sigma(a)$ does not contain any negative real number, then $a = \exp(b)$ for some $b \in A$. This fact is used in proving that $G_1 = \{\exp(a) : a \in A\}$. A proof of this given in [1] or [7] for complex Banach algebras works for real Banach algebras as well. (See also [5].)

Theorem 2.1. *Let A be a commutative real Banach algebra with unit* 1*. Let G denote the group of invertible elements of A and let G_1 be the connected component* *of G containing 1. If the quotient group G/G_1 contains an element of finite order other than G_1, then the order of such an element must be 2.*

Proof. It is enough to prove the following claim.
Claim: If $a \in G$ and $a^m \in G_1$ for some natural number m, then $a^2 \in G_1$.

Since $a^m \in G_1$, we have $a^m = \exp(x)$ for some $x \in A$. Let $b = \exp(x/m) \in G_1$ and $c = ab^{-1}$. Then by the commutativity of A, $c^m = a^m b^{-m} = \exp(x)\exp(-x) = 1$. Define $g : \mathbb{C} \to A$ by

$$g(\lambda) = (s^2+t^2)c^2 - 2(s^2+t^2-s)c + (s^2+t^2-2s+1), \quad \lambda = s+it \in \mathbb{C}.$$

Let $E = \{\lambda \in \mathbb{C} : g(\lambda) \in G\}$. Suppose $\lambda = s+it \notin E$. Then $\lambda \neq 0$ because $g(0) = 1 \in G$. Also since, $\lambda = s+it \notin E$, we have

$$g(\lambda) = (s^2+t^2)c^2 - 2(s^2+t^2-s)c + (s^2+t^2-2s+1) \notin G.$$

Since $s^2+t^2 \neq 0$, this gives

$$c^2 - 2(1 - \frac{s}{s^2+t^2})c + (1 - \frac{2s}{s^2+t^2} + \frac{1}{s^2+t^2}) \notin G.$$

A little calculation/simplification shows that this last expression is in fact $(c-\alpha)^2 + \beta^2 \notin G$, where $\alpha + i\beta = \frac{\lambda-1}{\lambda}$. In other words, $\frac{\lambda-1}{\lambda} \in \sigma(c)$. Hence $(\frac{\lambda-1}{\lambda})^m \in \sigma(c^m) = \sigma(1) = \{1\}$. This implies $(\lambda-1)^m = \lambda^m$. Since this last equation can have only a finite number of solutions in $\mathbb{C}$, $\mathbb{C}\setminus E$ is a finite set. Thus E is a connected set. Hence $g(E)$ is a connected subset of G containing $g(0) = 1$. Thus $g(E) \subseteq G_1$. In particular, $c^2 = g(1) \in G_1$ and in turn $a^2 = c^2b^2 \in G_1$. This proves the claim. □

Remark 2.2. Note that in view of the above theorem, if G/G_1 is a finite group, then every element other than the identity element G_1 is of order 2. Hence the group G/G_1 is isomorphic to $\mathbb{Z}_2^n$ for some nonnegative integer n. In particular, the order of G/G_1 is 2^n. This happens in the case of the real Banach algebra $\mathbb{R}^n$ with the coordinatewise multiplication and the max norm $\|.\|_\infty$.

Remark 2.3. We may further note that if A is not commutative, then G/G_1 can contain elements of finite order other than 1 or 2. In fact, Paulsen has shown in [6], that given a natural number n, it is possible to construct a Banach algebra A such that the group G/G_1 of this algebra A is (isomorphic to) the cyclic group of order n.

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