

*S. H. Kulkarni and G. Ramesh*

*On the denseness of minimum attaining operators*

# OPERATORS AND MATRICES

## AIMS AND SCOPE

'Operators and Matrices' (OaM), aims towards developing a high standard international journal which will publish top quality research papers in matrix and operator theory and their applications. The journal will publish mainly pure mathematics, but occasionally papers of a more applied nature could be accepted providing they have a nontrivial mathematical content. OaM will also publish expository papers and relevant book reviews.

OaM is published quarterly, in March, June, September and December.

## SUBMISSION

Manuscripts should be submitted through OaM page: [oam.ele-math.com](http://oam.ele-math.com), by e-mail of Editorial office, or directly to any one of the editors working in a related subject area.

Manuscripts are accepted for refereeing on the understanding that the same work has not been published (except in the form of an abstract), and is not under consideration for publication elsewhere.

Authors are advised to send notification of submission by e-mail to [oam@element.hr](mailto:oam@element.hr) (author(s), title, number of pages, name of editor to whom paper was sent to).

In order to facilitate refereeing, copies of those papers (whether by the author or someone else) which are essential and referred to in the manuscript but are not conveniently accessible, should be enclosed.

The publisher strongly encourages submission of manuscripts written in TeX or one of its variants LaTeX, AMSTeX or AMSLaTeX. On acceptance of the paper, authors will be asked to send the file by electronic mail. The file must be the one from which the accompanying manuscript (finalized version) was printed out. A poscript or pdf file of this final version must also be enclosed.

## TITLE PAGE

The following data should be included on title page: the title of the article, author's name (no degrees) author's affiliation, e-mail addresses, mailing address of the corresponding author, and running head less than 60 characters.

## ABSTRACT, KEY WORDS, SUBJECT CLASSIFICATION

The manuscript must be accompanied by a brief abstract, no longer than 100-150 words. It should make minimal use of mathematical symbols and displayed formulas. Mathematics Subject Classification (2010) and a list of 4-5 key words must be given.

## FIGURES

Figures should be prepared in a digital form suitable for direct reproduction, at resolution of 300 dpi or higher, and in EPS, TIFF or JPEG format.

## REFERENCES

Bibliographic references should be listed alphabetically at the end of the article. The authors should consult Mathematical Reviews for the standard abbreviations of journal names.

## PROOFS, PAGE CHARGES, OFFPRINTS

The galley proofs will be sent to corresponding author. Late return of the proofs will delay the article to a later issue.

There are no page charges. Authors will receive PDF file of the printed article free of charge. Additional offprints may be ordered from OaM prior to publication.

## FORTHCOMING PAPERS

Papers accepted and prepared for publication will appear in the forthcoming section of Journal Web page. They are identical in form as final printed papers, except volume, issue and page numbers.

OaM is published by Publishing House **ELEMENT**, Zagreb, Croatia.

The publication of this journal is supported by Ministry of science, education and sports, Croatia

All correspondence and subscription orders should be addressed to the Editorial Office:

[www.ele-math.com](http://www.ele-math.com)

e-mail: [oam@ele-math.com](mailto:oam@ele-math.com)

Operators and Matrices  
Editorial Office  
Menceticeva 2, 10000 Zagreb, Croatia  
Fax: +385 1 6008799

## ON THE DENSENESS OF MINIMUM ATTAINING OPERATORS

S. H. KULKARNI AND G. RAMESH

(Communicated by D. R. Farenick)

*Abstract.* Let  $H_1, H_2$  be complex Hilbert spaces and  $T$  be a densely defined closed linear operator (not necessarily bounded). It is proved that for each  $\varepsilon > 0$ , there exists a bounded operator  $S$  with  $\|S\| \leq \varepsilon$  such that  $T + S$  is minimum attaining. Further, if  $T$  is bounded below, that is if there exists  $m > 0$  such that  $\|Tx\| \geq m\|x\|$  for every  $x$  in the domain of  $T$ , then  $S$  can be chosen to be rank one.

### 1. Introduction

It is well known that the set of all norm attaining operators defined between two complex Hilbert spaces is norm dense in the space of all bounded linear operators defined between complex Hilbert spaces. This result is even true for operators defined between Banach spaces, when the domain space is reflexive, which is proved by Lindenstrauss [14]. A simple proof of this fact, in the case of Hilbert space operators is given by Enflo et al. in [6]. Moreover, the authors proved that rank one perturbation of a bounded operator can be made as norm attaining operator.

Similar to the norm attaining operators, bounded operators that attain their minimum modulus is introduced in [5]. The unbounded case is dealt in [13] and the authors established basic properties of minimum attaining closed densely defined operators.

It is very natural to ask whether the Lindenstrauss theorem is true in case of minimum attaining operators. In this article we answer this question affirmatively. We show that the set of all minimum attaining densely defined closed operators is dense in the class of densely defined closed operators with respect to the gap metric. As a consequence, we can conclude that the same is true for bounded operators with respect to the operator norm. In a special case, we also show that rank one perturbations lead to minimum attaining operators. This leads to the perturbations of minimum attaining operators.

In the second section we recall some basic definitions and results which we need for proving our main results. In the third section we prove the Lindenstrauss type theorem for minimum attaining operators.

---

*Mathematics subject classification* (2010): 47A05, 47A55.

*Keywords and phrases:* Densely defined closed operator, minimum attaining operator, gap metric, Weyl's theorem, Lindenstrauss theorem.

## 2. Notations and preliminaries

Throughout we consider infinite dimensional complex Hilbert spaces which will be denoted by  $H, H_1, H_2$  etc. The inner product and the induced norm are denoted by  $\langle \cdot \rangle$  and  $\|\cdot\|$ , respectively. The closure of a subspace  $M$  of  $H$  is denoted by  $\overline{M}$ . We denote the unit sphere of  $M$  by  $S_M = \{x \in M : \|x\| = 1\}$ .

If  $M$  is a closed subspace of a Hilbert space  $H$ , then  $P_M$  denotes the orthogonal projection  $P_M : H \rightarrow H$  with range  $M$ .

Let  $T$  be a linear operator with domain  $D(T)$  (a subspace of  $H_1$ ) and taking values in  $H_2$ . If  $D(T)$  is dense in  $H_1$ , then  $T$  is called a *densely defined* operator. The graph  $G(T)$  of  $T$  is defined by  $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$ . If  $G(T)$  is closed, then  $T$  is called a *closed operator*. Equivalently,  $T$  is closed if  $(x_n)$  is a sequence in  $D(T)$  such that  $x_n \rightarrow x \in H_1$  and  $Tx_n \rightarrow y \in H_2$ , then  $x \in D(T)$  and  $Tx = y$ .

For a densely defined linear operator  $T$ , there exists a unique linear operator (in fact, a closed operator)  $T^* : D(T^*) \rightarrow H_1$ , with

$$D(T^*) := \{y \in H_2 : x \mapsto \langle Tx, y \rangle \text{ for all } x \in D(T) \text{ is continuous}\} \subseteq H_2$$

satisfying

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \text{ for all } x \in D(T) \text{ and } y \in D(T^*).$$

We say  $T$  to be *bounded* if there exists  $M > 0$  such that  $\|Tx\| \leq M\|x\|$  for all  $x \in D(T)$ . Note that if  $T$  is densely defined and bounded then  $T$  can be extended to whole of  $H_1$  in a unique way.

By the closed graph Theorem, an everywhere defined closed operator is bounded. Hence the domain of an unbounded closed operator is a proper subspace of a Hilbert space.

The space of all bounded linear operators between  $H_1$  and  $H_2$  is denoted by  $\mathcal{B}(H_1, H_2)$  and the class of all closed linear operators between  $H_1$  and  $H_2$  is denoted by  $\mathcal{C}(H_1, H_2)$ . We write  $\mathcal{B}(H, H) = \mathcal{B}(H)$  and  $\mathcal{C}(H, H) = \mathcal{C}(H)$ .

If  $T \in \mathcal{B}(H_1, H_2)$  is such that for every bounded sequence  $(x_n)$  of  $H_1$ ,  $(Tx_n)$  has a convergent subsequence in  $H_2$ , then  $T$  is called a *compact operator*. Equivalently,  $T$  is compact if and only if for every bounded set  $B$  of  $H_1$ ,  $T(B)$  is pre compact in  $H_2$ .

If  $T \in \mathcal{C}(H_1, H_2)$ , then the null space and the range space of  $T$  are denoted by  $N(T)$  and  $R(T)$ , respectively and the space  $C(T) := D(T) \cap N(T)^\perp$  is called the carrier of  $T$ . In fact,  $D(T) = N(T) \oplus^\perp C(T)$  [3, Page 340]. Here  $\oplus^\perp$  denote the orthogonal direct sum of subspaces.

Let  $S, T \in \mathcal{C}(H)$  be densely defined operators with domains  $D(S)$  and  $D(T)$ , respectively. Then  $S + T$  is an operator with domain  $D(S + T) = D(S) \cap D(T)$  defined by  $(S + T)(x) = Sx + Tx$  for all  $x \in D(S + T)$ . The operator  $ST$  has the domain  $D(ST) = \{x \in D(T) : Tx \in D(S)\}$  and is defined as  $(ST)(x) = S(Tx)$  for all  $x \in D(ST)$ .

If  $S$  and  $T$  are closed operators with the property that  $D(T) \subseteq D(S)$  and  $Tx = Sx$  for all  $x \in D(T)$ , then  $S$  is called the *restriction* of  $T$  and  $T$  is called an *extension* of  $S$ .

A densely defined operator  $T \in \mathcal{C}(H)$  is said to be

1. *normal* if  $T^*T = TT^*$
2. *self-adjoint* if  $T = T^*$
3. *positive* if  $T = T^*$  and  $\langle Tx, x \rangle \geq 0$  for all  $x \in D(T)$ .

Let  $V \in \mathcal{B}(H_1, H_2)$ . Then  $V$  is called

1. an *isometry* if  $\|Vx\| = \|x\|$  for all  $x \in H_1$
2. a *partial isometry* if  $V|_{N(V)^\perp}$  is an isometry. The space  $N(V)^\perp$  is called the *initial space* or the *initial domain* and the space  $R(V)$  is called the *final space* or the *final domain* of  $V$ .

DEFINITION 2.1. Let  $T \in \mathcal{B}(H_1, H_2)$ . Then  $T$  is said to be *norm attaining* if there exists  $x_0 \in S_{H_1} \subseteq H_1$  such that  $\|Tx_0\| = \|T\|$ .

An analogous concept to the norm of an operator is the minimum modulus of an operator. This concept was introduced in [7].

DEFINITION 2.2. [7] Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then

$$m(T) = \inf \{ \|Tx\| : x \in S_{D(T)} \}$$

is called the *minimum modulus* of  $T$ .

DEFINITION 2.3. Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then  $T$  is said to be *bounded below* if there exists  $m > 0$  such that  $\|Tx\| \geq m\|x\|$  for all  $x \in D(T)$ .

REMARK 2.4.

1. This definition of bounded below operators is analogous to the usual definition given in Functional Analysis books (for example [1, Definition 2.1, Page 69] ) for the case of bounded operators.
2. It is easy to see that  $T$  is bounded below if and only if  $m(T) > 0$ . In this case  $m(T) = \sup \{ m > 0 : \|Tx\| \geq m\|x\|, \text{ for all } x \in D(T) \}$ .
3.  $T$  is bounded below if and only if  $R(T)$  is closed and  $T$  is one-to-one.

If  $T \in \mathcal{C}(H_1, H_2)$  is densely defined and one-to-one, then the *inverse operator* is the linear operator from  $H_2$  into  $H_1$  with  $D(T^{-1}) = R(T)$  and  $T^{-1}Tx = x$  for all  $x \in D(T)$ . In this case we say  $T$  is *invertible* and  $R(T^{-1}) = D(T)$  (see [20, Page 4] for details). In particular, if  $T$  is one-to-one and onto, then  $T^{-1}$  is defined on whole of  $H_2$  and is continuous by the closed graph theorem.

Note that if  $T \in \mathcal{C}(H)$  is normal, then  $T$  has a bounded inverse if and only if  $m(T) > 0$ .

**DEFINITION 2.5.** [13, Definition 3.1] Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then we say  $T$  to be *minimum attaining* if there exists  $x_0 \in S_{D(T)}$  such that  $\|Tx_0\| = m(T)$ .

We denote the class of minimum attaining densely defined closed operators between  $H_1$  and  $H_2$  by  $\mathcal{M}_c(H_1, H_2)$  and  $\mathcal{M}_c(H, H)$  by  $\mathcal{M}_c(H)$ .

**NOTE 2.6.** If  $T \in \mathcal{C}(H_1, H_2)$  is densely defined and  $N(T) \neq \{0\}$ , then  $m(T) = 0$  and there exists  $x \in S_{N(T)}$  such that  $Tx = 0$ . Hence  $T \in \mathcal{M}_c(H_1, H_2)$ .

**THEOREM 2.7.** [19, Theorem 13.31, Page 369][4, Theorem 4, Page 144] Let  $T \in \mathcal{C}(H)$  be densely defined and positive. Then there exists a unique positive operator  $S$  such that  $T = S^2$ . The operator  $S$  is called the square root of  $T$  and is denoted by  $S = T^{\frac{1}{2}}$ .

For  $T \in \mathcal{C}(H_1, H_2)$  densely defined, the operator  $|T| := (T^*T)^{\frac{1}{2}}$  is called the modulus of  $T$ . Moreover,  $D(|T|) = D(T)$ ,  $N(|T|) = N(T)$  and  $\overline{R(|T|)} = \overline{R(T^*)}$ . As  $\|Tx\| = \||T|x\|$  for all  $x \in D(T)$ , we can conclude that  $m(T) = m(|T|)$  and  $T \in \mathcal{M}_c(H_1, H_2)$  if and only if  $|T| \in \mathcal{M}_c(H_1)$ .

**THEOREM 2.8.** [4, Theorem 2, Page 184] Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then there exists a unique partial isometry  $V : H_1 \rightarrow H_2$  with the initial space  $\overline{R(T^*)}$  and the final space  $\overline{R(T)}$  such that  $T = V|T|$ .

**DEFINITION 2.9.** [19, Definition 13.26, Theorem 13.27, Page 365–366] Let  $T \in \mathcal{C}(H)$  be densely defined. The resolvent of  $T$  is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \rightarrow H \text{ is invertible and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

and

$$\begin{aligned} \sigma(T) &:= \mathbb{C} \setminus \rho(T) \\ \sigma_p(T) &:= \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \rightarrow H \text{ is not one-to-one}\}, \end{aligned}$$

are called the spectrum and the point spectrum of  $T$ , respectively.

A formula to compute the minimum modulus of a densely defined closed operator is given as follows.

**PROPOSITION 2.10.** [13, Proposition 3.3] Let  $T \in \mathcal{C}(H)$  be normal. Then

$$m(T) = d(0, \sigma(T)),$$

where  $d(x, A)$  denote the distance between the point  $x$  and the set  $A$ .

REMARK 2.11. If  $T \in \mathcal{C}(H)$  is normal, then  $\sigma(T)$  is closed (the details can be found in [20, Proposition 2.6(ii), Page 29]). Hence by Proposition (2.10), it is clear that there exists a  $\lambda \in \sigma(T)$  such that  $|\lambda| = m(T)$ .

DEFINITION 2.12. [20, Definition 8.3 Page 178] Let  $T = T^* \in \mathcal{C}(H)$ . Then the *discrete spectrum*  $\sigma_d(T)$  of  $T$  is defined as the set of all eigenvalues of  $T$  with finite multiplicities which are isolated points of the spectrum  $\sigma(T)$  of  $T$ . The complement set  $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_d(T)$  is called the *essential spectrum* of  $T$ .

The essential spectrum is stable under compact perturbations.

PROPOSITION 2.13. (Weyl’s theorem) [20, Corollary 8.16, Page 182] Let  $T \in \mathcal{C}(H)$  be self-adjoint and  $C$  is compact, self-adjoint. Then  $\sigma_{ess}(T + C) = \sigma_{ess}(T)$ .

LEMMA 2.14. [8, 9, 16] Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Denote  $\check{T} = (I + T^*T)^{-1}$  and  $\hat{T} = (I + TT^*)^{-1}$ . Then

1.  $\check{T} \in \mathcal{B}(H_1)$ ,  $\hat{T} \in \mathcal{B}(H_2)$
2.  $\hat{T}T \subseteq T\check{T}$ ,  $\|T\check{T}\| \leq \frac{1}{2}$  and  $\check{T}T^* \subseteq T^*\hat{T}$ ,  $\|T^*\hat{T}\| \leq \frac{1}{2}$
3. if  $g : [0, 1] \rightarrow \mathbb{C}$  is a continuous function, then
  - (a)  $T^*g(\hat{T})y = g(\check{T})T^*y$  for all  $y \in D(T^*)$
  - (b)  $Tg(\check{T})x = g(\hat{T})Tx$  for all  $x \in D(T)$ .

### 2.1. Gap metric

The gap between closed subspaces  $M$  and  $N$  of  $H$  is defined by  $\theta(M, N) := \|P_M - P_N\|$ . This defines a metric on the class of closed subspaces of  $H$ , known as the gap metric. The topology induced by the gap metric is known as the gap topology. We have the following alternative formula for the gap;

$$\theta(M, N) := \max \left\{ \|P_M(I - P_N)\|, \|P_N(I - P_M)\| \right\}.$$

For the details we refer to [2, Page 70].

Let  $A, B \in \mathcal{C}(H_1, H_2)$  be densely defined. Then  $G(A)$  and  $G(B)$  are closed subspaces of  $H_1 \times H_2$ . The gap between  $A$  and  $B$  is defined as

$$\theta(A, B) = \|P_{G(A)} - P_{G(B)}\|,$$

where  $P_M : H_1 \times H_2 \rightarrow H_1 \times H_2$ , is an orthogonal projection onto the closed subspace  $M$  of  $H_1 \times H_2$ . This defines a metric on the class of closed operators and induced topology is known as the gap topology. The gap topology restricted to the space of bounded linear operators coincides with the norm topology. Also the convergence with respect to the gap metric on the set of self-adjoint bounded operators coincide with the resolvent convergence [18, Chapter VII, Page 235]).

We have the following formula for the gap between two closed operators;



**THEOREM 2.15.** [12, Theorem 3.5] *Let  $S, T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then the operators  $\widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}$ ,  $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}$ ,  $S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}}$  and  $\widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}}$  are bounded and*

$$\theta(S, T) = \max \left\{ \left\| \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}} - T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}} \right\|, \left\| S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} - \widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}} \right\| \right\}. \quad (2.1)$$

### 3. Denseness of minimum attaining operators

In this section we discuss the denseness of minimum attaining operators. First let us consider the case of functionals:

Let  $H$  be a Hilbert space and  $\phi : H \rightarrow \mathbb{C}$  be a non zero linear functional. Then  $\phi$  is continuous (bounded) if and only if  $\phi$  is closed. Since  $H$  is infinite dimensional and  $H/N(\phi)$  is isomorphic with  $\mathbb{C}$ , we can clearly conclude that  $N(\phi) \neq \{0\}$ . Hence  $\phi$  is minimum attaining. Thus, the class of minimum attaining bounded linear functionals coincide with the space of all bounded linear functionals. So in this case, the minimum attaining functionals are dense. If  $H$  is finite dimensional, then clearly every linear functional is minimum attaining. Hence in this case also the result holds trivially.

In the above discussion we can replace Hilbert space by a Banach space. By a theorem of James we can conclude that a normed linear space  $X$  is reflexive if and only if every non zero bounded linear functional is norm attaining (see [10, 11] for details). This is no more true if we replace the norm attaining property of functionals by minimum attaining property, as we have noted in the above paragraph.

Now we consider the case of densely defined closed operators defined between two different Hilbert spaces. We prove that the set of all minimum attaining densely defined closed operators defined between two Hilbert spaces is dense in the class of all densely defined closed operators with respect to the gap metric. To prove this we need the following result, which is proved in [17, Remark 3.7, Corollary 3.9] for regular operators defined between two Hilbert  $C^*$ -modules. For the sake of completeness, we provide the details here.

**THEOREM 3.1.** *Let  $S, T \in \mathcal{C}(H_1, H_2)$  be densely defined and  $D(S) = D(T)$ . Then*

1. *the operators  $\widehat{T}^{\frac{1}{2}}(T - S)\check{S}^{\frac{1}{2}}$  and  $\widehat{S}^{\frac{1}{2}}(S - T)\check{T}^{\frac{1}{2}}$  are bounded and*

$$\theta(S, T) = \max \left\{ \left\| \widehat{T}^{\frac{1}{2}}(T - S)\check{S}^{\frac{1}{2}} \right\|, \left\| \widehat{S}^{\frac{1}{2}}(S - T)\check{T}^{\frac{1}{2}} \right\| \right\}$$

2. *if  $T - S$  is bounded, then  $\theta(S, T) \leq \|S - T\|$ .*

*Proof.* First we simplify the term  $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}$ . For  $x \in H_1$ , we have that  $\check{S}^{\frac{1}{2}}x \in D(S) = D(T)$ , consequently,  $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}x = \widehat{T}^{\frac{1}{2}}T\check{S}^{\frac{1}{2}}x$ , by (3) of Lemma 2.14. Thus,

$$T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}x - \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}x = \left( \widehat{T}^{\frac{1}{2}}T\check{S}^{\frac{1}{2}} - \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}} \right) x \quad (3.1)$$

$$= \widehat{T}^{\frac{1}{2}}(T - S)\check{S}^{\frac{1}{2}}x. \quad (3.2)$$



With a similar argument we can show that

$$S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} - \widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}} = \widehat{S}^{\frac{1}{2}}(S - T)\check{T}^{\frac{1}{2}}.$$

Now the conclusion follows by Theorem 2.15.

If  $T - S$  is bounded, then

$$\|\widehat{T}^{\frac{1}{2}}(T - S)\check{S}^{\frac{1}{2}}\| \leq \|\widehat{T}^{\frac{1}{2}}\| \|T - S\| \|\check{S}^{\frac{1}{2}}\| \leq \|T - S\|. \tag{3.3}$$

Similarly, we can conclude that  $\|\widehat{S}^{\frac{1}{2}}(S - T)\check{T}^{\frac{1}{2}}\| \leq \|T - S\|$ .

Hence by the above two observations the conclusion follows.  $\square$

PROPOSITION 3.2. [13, Propositions 3.8, 3.9] *Let  $T \in \mathcal{C}(H)$  be positive. Then*

1.  $T \in \mathcal{M}_c(H)$  if and only if  $T^{\frac{1}{2}} \in \mathcal{M}_c(H)$
2.  $T \in \mathcal{M}_c(H)$  if and only if  $m(T)$  is an eigenvalue of  $T$ .

PROPOSITION 3.3. [13, Proposition 3.5] *Let  $T \in \mathcal{C}(H)$  be positive. Then*

$$m(T) = \inf \{ \langle Tx, x \rangle : x \in S_{D(T)} \}.$$

THEOREM 3.4. *Let  $T \in \mathcal{C}(H)$  be positive. Then for each  $\varepsilon > 0$ , there exists an operator  $S \in \mathcal{B}(H)$  such that*

1.  $\|S\| \leq \varepsilon$
2.  $T + S$  is minimum attaining
3.  $\theta(S + T, T) \leq \varepsilon$ .

Moreover, if  $m(T) > 0$ , then  $S$  can be chosen such that  $\text{rank}(S) = 1$  and  $m(T + S) > 0$ .

*Proof.* We prove the results by considering the following three cases which exhaust all possibilities.

Case (1):  $m(T) > 0$

It suffices to prove the assertion for  $\varepsilon \in (0, m(T))$ . Since,  $T \geq 0$  and  $m(T) = \inf_{x \in S_{D(T)}} \langle Tx, x \rangle$  by Proposition 3.3, there exists  $x_\varepsilon \in S_{D(T)}$ , such that

$$\langle Tx_\varepsilon, x_\varepsilon \rangle < m(T) + \frac{\varepsilon}{2}. \tag{3.4}$$

Now, define

$$C_\varepsilon(x) = \varepsilon \langle x, x_\varepsilon \rangle x_\varepsilon, \text{ for every } x \in H. \tag{3.5}$$

Then clearly,  $C_\varepsilon$  is a rank one positive, bounded operator with  $\|C_\varepsilon\| = \|C_\varepsilon(x_\varepsilon)\| = \varepsilon$ .

Let  $T_\varepsilon = T - C_\varepsilon$ . It is easy to see that  $T_\varepsilon$  is a closed densely defined operator with  $D(T_\varepsilon) = D(T)$ . Clearly,  $T_\varepsilon$  is self-adjoint. In fact, we show that  $T_\varepsilon \geq 0$ . To this end, let  $x \in D(T_\varepsilon) = D(T)$ . Then

$$\begin{aligned} \langle T_\varepsilon x, x \rangle &= \langle Tx, x \rangle - \varepsilon |\langle x, x_\varepsilon \rangle|^2 \\ &\geq (m(T) - \varepsilon) \langle x, x \rangle \quad (\text{by Cauchy-Schwarz inequality}). \end{aligned}$$

From this we can conclude that  $T_\varepsilon \geq 0$  and by Proposition 3.3, we have that  $m(T_\varepsilon) \geq m(T) - \varepsilon > 0$ .

We claim that  $T_\varepsilon \in \mathcal{M}_c(H)$ . By Remark 2.11, we have that  $m(T_\varepsilon) \in \sigma(T_\varepsilon)$ . Next, we show that  $m(T_\varepsilon) \in \sigma_d(T_\varepsilon)$ . Assume that  $m(T_\varepsilon) \in \sigma_{ess}(T_\varepsilon)$ . Then by the Weyl's theorem we have  $\sigma_{ess}(T_\varepsilon) = \sigma_{ess}(T)$ . Note as  $m(T) \in \sigma(T)$  and  $m(T)$  is the smallest spectral value, we can conclude that  $m(T) \leq m(T_\varepsilon)$ .

But we have

$$m(T_\varepsilon) \leq \langle T_\varepsilon x_\varepsilon, x_\varepsilon \rangle = \langle Tx_\varepsilon, x_\varepsilon \rangle - \varepsilon < m(T) - \frac{\varepsilon}{2} < m(T).$$

Thus our assumption that  $m(T_\varepsilon) \in \sigma_{ess}(T_\varepsilon)$  is wrong. Consequently  $m(T_\varepsilon) \in \sigma_d(T_\varepsilon)$ , and hence  $T_\varepsilon \in \mathcal{M}_c(H)$ .

Note that  $T_\varepsilon - T = -C_\varepsilon|_{D(T)}$  is a bounded operator with domain  $D(T)$ . By Theorem 3.1, it follows that

$$\theta(T_\varepsilon, T) \leq \|T_\varepsilon - T\| = \|C_\varepsilon|_{D(T)}\| \leq \|C_\varepsilon\| = \varepsilon.$$

Take  $S = -C_\varepsilon$ . Then  $S$  satisfies the stated conditions. Note that  $m(T + S) = m(T_\varepsilon) > 0$ .

*Case (2):*  $m(T) = 0$  and  $T$  is not one-to-one

Clearly  $T$  is minimum attaining. In this case  $S = 0$  satisfy the required properties.

*Case (3):*  $m(T) = 0$  and  $T$  is one-to-one

We can use case (1) to get the desired operator  $S$ . Note that  $T + \frac{\varepsilon}{2}I$  is positive and  $m\left(T + \frac{\varepsilon}{2}I\right) = \frac{\varepsilon}{2}$ . Hence by Case (1) above, there exists a positive rank one operator  $C$  with  $\|C\| \leq \frac{\varepsilon}{2}$  such that  $T + \frac{\varepsilon}{2}I - C$  is minimum attaining and  $\theta\left(T, T + \frac{\varepsilon}{2}I - C\right) \leq \left\|\frac{\varepsilon}{2}I - C\right\| \leq \varepsilon$ . Let  $S = \frac{\varepsilon}{2}I - C$ . Then  $S$  satisfies all the required conditions.  $\square$

Now we prove the above result for the general case.

**THEOREM 3.5.** *Let  $T \in \mathcal{C}(H_1, H_2)$  be densely defined. Then for each  $\varepsilon > 0$  there exists an operator  $S \in \mathcal{B}(H_1, H_2)$  with  $\|S\| \leq \varepsilon$  such that  $T + S$  is minimum attaining and  $\theta(T + S, T) \leq \varepsilon$ . Moreover, if  $m(T) > 0$  then  $S$  can be chosen such that  $\text{rank}(S) = 1$  and  $m(T + S) > 0$ .*

*Proof.* By Note 2.6, it suffices to consider the case for operators that are injective. Let  $T = V|T|$  be the polar decomposition of  $T$ . Applying Theorem 3.4 to  $|T|$ , there exists  $A \in \mathcal{B}(H_1)$  with  $\|A\| \leq \varepsilon$  and  $|T| + A$  is minimum attaining.

Define  $S = VA$ . Then  $S \in \mathcal{B}(H_1, H_2)$  with  $\|S\| \leq \varepsilon$ . Next, we claim that  $T + S = V(|T| + A)$  is minimum attaining. Since  $T$  is one-to-one,  $V$  is an isometry and  $T + S$  is minimum attaining as  $|T| + A$  minimum attaining. Note that  $m(T + S) = m(|T| + A)$ .

In case if  $m(T) > 0$ , then by Theorem 3.4,  $A$  can be chosen so that  $\text{rank}(A) = 1$  and  $m(|T| + A) > 0$ . Hence  $\text{rank}(S) = \text{rank}(A) = 1$  and  $m(T + S) = m(|T| + A) > 0$ .

Finally, by Theorem 3.1, we have  $\theta(T + S, T) \leq \|S\| \leq \varepsilon$ .  $\square$

The following Corollary is an immediate consequence of Theorem 3.5.

**COROLLARY 3.6.** *The set  $\mathcal{M}_c(H_1, H_2)$  is dense in the set of all densely defined operators in  $\mathcal{C}(H_1, H_2)$  with respect to the gap topology.*

**COROLLARY 3.7.** *The set of all minimum attaining bounded operators is dense in  $\mathcal{B}(H_1, H_2)$  with respect to the norm topology of  $\mathcal{B}(H_1, H_2)$ .*

*Proof.* The gap topology restricted to  $\mathcal{B}(H_1, H_2)$  coincide with the norm topology of  $\mathcal{B}(H_1, H_2)$  by [15, Theorem 2.5]. Hence the conclusion follows by Corollary 3.6.  $\square$

**THEOREM 3.8.** *Let*

$$\mathcal{G} := \{T \in \mathcal{M}_c(H_1, H_2) : T \text{ is bounded below} \}$$

*and  $C_b(H_1, H_2) = \{T \in \mathcal{C}(H_1, H_2) : T \text{ is densely defined and bounded below}\}$ . If  $T \in C_b(H_1, H_2)$  and  $\varepsilon > 0$ , there exists  $\tilde{T} \in \mathcal{G}$  such that  $\theta(T, \tilde{T}) \leq \varepsilon$ . In other words,  $\mathcal{G}$  is dense in  $C_b(H_1, H_2)$  with respect to the gap metric.*

*Proof.* Let  $T \in C_b(H_1, H_2)$ . Since  $T$  is bounded below  $m(T) > 0$ . Hence by Theorem 3.5, there exists  $S \in \mathcal{B}(H_1, H_2)$  such that  $T + S$  is minimum attaining,  $\theta(T + S, T) \leq \varepsilon$ ,  $\text{rank}(S) = 1$  and  $m(T + S) > 0$ , that is  $T + S$  is bounded below. Take  $\tilde{T} = T + S$ . Then  $\tilde{T}$  satisfy all the requirements.  $\square$

Recall that  $A \in \mathcal{B}(H_1, H_2)$  is bounded below if there exists  $k > 0$  such that  $\|Ax\| \geq k\|x\|$  for all  $x \in H_1$ . In this case, the minimum modulus

$$m(A) := \inf \{ \|Ax\| : x \in S_{H_1} \}$$

is given by  $m(A) = \sup \{ k > 0 : \|Ax\| \geq k\|x\|, \text{ for all } x \in H_1 \}$ .

**COROLLARY 3.9.** *The set*

$$\mathcal{M}_b(H_1, H_2) := \{A \in \mathcal{B}(H_1, H_2) : A \text{ is minimum attaining and bounded below}\}$$

*is norm dense in  $\mathcal{B}_b(H_1, H_2) = \{A \in \mathcal{B}(H_1, H_2) : A \text{ is bounded below}\}$ .*

*Proof.* The gap metric and the metric induced by the operator norm are equivalent on  $\mathcal{B}(H_1, H_2)$  by [15, Theorem 2.5]. Hence the conclusion follows by Theorem 3.8.  $\square$

*Acknowledgements.* The authors would like to thank the referee for his critical comments which improved the readability of the paper and the handling editor for the help during the editorial process.

#### REFERENCES

- [1] Y. A. ABRAMOVICH AND C. D. ALIPRANTIS, *An invitation to operator theory*, Graduate Studies in Mathematics **50**, American Mathematical Society, Providence, RI, 2002, MR 1921782.
- [2] N. I. AKHIEZER AND I. M. GLAZMAN, *Theory of linear operators in Hilbert space*, Dover Publications, Inc., New York, 1993, Translated from the Russian and with a preface by Merlynd Nestell, Reprint of the 1961 and 1963 translations, Two volumes bound as one, MR 1255973.
- [3] ADI BEN-ISRAEL AND THOMAS N. E. GREVILLE, *Generalized inverses: theory and applications*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974, Pure and Applied Mathematics, MR 0396607.
- [4] M. SH. BIRMAN AND M. Z. SOLOMJAK, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), Translated from the 1980 Russian original by S. Khrushchëv and V. Peller, D. Reidel Publishing Co., Dordrecht, 1987, MR 1192782.
- [5] XAVIER CARVAJAL AND WLADIMIR NEVES, *Operators that attain their minima*, Bull. Braz. Math. Soc. (N.S.) **45** (2014), no. 2, 293–312, MR 3249529.
- [6] PER ENFLO, JANICE KOVER AND LAURA SMITHIES, *Denseness for norm attaining operator-valued functions*, Linear Algebra Appl. **338** (2001), 139–144, MR 1861118.
- [7] H. A. GINDLER AND A. E. TAYLOR, *The minimum modulus of a linear operator and its use in spectral theory*, Studia Math., Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica **22**, MR 0151850.
- [8] SIMONE GRAMSCH AND EBERHARD SCHOCK, *Ill-posed equations with transformed argument*, Abstr. Appl. Anal. (2003), no. 13, 785–791, MR 1996924.
- [9] C. W. GROETSCH, *Spectral methods for linear inverse problems with unbounded operators*, J. Approx. Theory **70** (1992), no. 1, 16–28, MR 1168372.
- [10] ROBERT C. JAMES, *Reflexivity and the supremum of linear functionals*, Ann. of Math. (2) **66** (1957), 159–169, MR 0090019.
- [11] ROBERT C. JAMES, *Reflexivity and the sup of linear functionals*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), vol. 13, 1972, pp. 289–300 (1973), MR 0338742.
- [12] S. H. KULKARNI AND G. RAMESH, *A formula for gap between two closed operators*, Linear Algebra Appl. **432** (2010), no. 11, 3012–3017, MR 2639265.
- [13] S. H. KULKARNI AND G. RAMESH, *Absolutely minimum attaining closed operators*, Preprint, <http://arxiv.org/abs/1606.05736> (2016).
- [14] JORAM LINDENSTRAUSS, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148, MR 0160094.
- [15] RITSUO NAKAMOTO, *Gap formulas of operators and their applications*, Math. Japon. **42** (1995), no. 2, 219–232, MR 1356379.
- [16] GERT K. PEDERSEN, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989, MR 971256.
- [17] G. RAMESH, *McIntosh formula for the gap between regular operators*, Banach J. Math. Anal. **7**, 2013, 97–106, MR 3004269.
- [18] MICHAEL REED AND BARRY SIMON, *Methods of modern mathematical physics. I*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis, MR 751959.
- [19] WALTER RUDIN, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991, MR 1157815.

- [20] KONRAD SCHMÜDGEN, *Unbounded self-adjoint operators on Hilbert space*, Graduate Texts in Mathematics, vol. 265, Springer, Dordrecht, 2012, MR 2953553.

(Received September 27, 2016)

*S. H. Kulkarni*  
*Department of Mathematics*  
*Indian Institute of Technology Madras*  
*Taramani, Chennai, Tamilnadu 600 036*  
*e-mail: shk@iitm.ac.in*

*G. Ramesh*  
*Department of Mathematics*  
*Indian Institute of Technology – Hyderabad*  
*Kandi, Sangareddy, Telangana, India 502 285*  
*e-mail: rameshg@iith.ac.in*

