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The null space theorem

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A R T I C L E I N F O

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The following results are proved.

Theorem 0.1 *(The Null Space Theorem). Let X, Y be vector spaces, $P \in L(X), Q \in L(Y)$ be projections and $T \in L(X, Y)$ be invertible. (The restriction of QTP to $R(P)$ can be viewed as a linear operator from $R(P)$ to $R(Q)$. This is called a* section of T by P and Q *and will be denoted by $T_{P,Q}$.) Then there is a linear bijection between the null space of the section $T_{P,Q}$ of T and the null space of its complementary section $T^{-1}_{I_Y - Q, I_X - P}$ of T^{-1}.*

Theorem 0.2. *Let X be a Banach space with a Schauder basis $A = \{a_1, a_2, \ldots\}$. Let T be a bounded (continuous) linear operator on X. Suppose the matrix of T with respect to A is tridiagonal. If T is invertible, then every submatrix of the matrix of T^{-1} with respect to A that lies on or above the main diagonal (or on or below the main diagonal) is of rank ≤ 1.*

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1. Introduction

This note has two objectives. The first is to make the Nullity Theorem known more widely. The second is to consider its generalization to infinite dimensional spaces and some applications of this generalization. We begin with some motivation for the Nullity Theorem.

Recall that a square matrix $A = [\alpha_{ij}]$ of order n is called *tridiagonal* if

$$\alpha_{ij} = 0 \quad \text{for } |i - j| > 1$$

Such a matrix is described completely by $3n - 2$ numbers (n on the main diagonal and $n - 1$ on each of superdiagonal and subdiagonal). In general, if a tridiagonal matrix is invertible, its inverse need not be tridiagonal. However, we may still expect that the inverse can be described completely by $3n-2$ parameters. This is indeed true. It is known that if A is a tridiagonal matrix of order n and if A is also invertible, then every submatrix of A^{-1} that lies on or above the main diagonal is of rank ≤ 1. Similar statement is true of submatrices lying on or below the main diagonal. This result is known at least since 1979 (see [2]). Several proofs of this result are available in the literature. The article [7] contains some of these proofs, references to these and other proofs and also a brief history and comments about possible generalizations.

In view of this result, the inverse can be described using $3n - 2$ parameters as follows: To start with we can choose $4n$ numbers a_j, b_j, c_j, d_j, $j = 1, \dots, n$ such that

$$\begin{aligned} \left(A^{-1}\right)_{ij} &= a_i b_j \quad \text{for } i \leq j \quad \text{and} \\ &= c_i d_j \quad \text{for } j \leq i \end{aligned}$$

These $4n$ numbers have to satisfy the following constraints

$$a_i b_i = c_i d_i \quad \text{for } i = 1, \dots, n \text{ and } a_1 = 1 = b_1.$$

One proof of this theorem depends on the Nullity Theorem. This theorem uses the idea of complementary submatrices. Let A and B be square matrices of order n. Suppose M is a submatrix of A and N is a submatrix of B. We say that M and N are *complementary* if row numbers not used in one are the column numbers used in the other. More precisely, let I and J be subsets of the set $\{1, 2, \dots, n\}$ and let I^c denote the complement of I. Let $A(I, J)$ denote the submatrix of A obtained by choosing rows in I and columns in J. Then $A(I, J)$ and $B(J^c, I^c)$ are complementary submatrices. With this terminology, the Nullity Theorem has a very simple formulation.

Theorem 1.1 *(Nullity Theorem). Complementary submatrices of a square matrix and its inverse have the same nullity.*

As an illustration we can consider the following. Suppose $k < n$ and a square matrix M of order n is partitioned into submatrices as follows:

$$M = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

Here A_k is the submatrix obtained from A by choosing the first k rows and the first k columns. Assume that M is invertible and its inverse is partitioned similarly as follows:

$$M^{-1} = \begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix}$$

Then the Nullity Theorem says that

$$\text{nullity}(A_k) = \text{nullity}(S_k), \qquad \text{nullity}(D_k) = \text{nullity}(P_k)$$
$$\text{nullity}(B_k) = \text{nullity}(Q_k), \qquad \text{nullity}(C_k) = \text{nullity}(R_k)$$

This Nullity Theorem has been in the literature for quite some time (at least since 1984), but it does not seem to be that widely well known. In [7], Gilbert Strang and Tri Ngyuen have given an account of this Nullity Theorem. They have given a proof of this theorem and discussed its consequences for ranks of some submatrices. In particular, they prove a very interesting fact that the submatrices of a banded invertible matrix lying above or below the main diagonal have low ranks. While discussing literature and alternative proofs, the authors make the following remark.

"A key question will be the generalization to infinite dimensions".

We attempt such a generalization in this note. It is called the "Null Space Theorem".

We recall a few standard notations, definitions and results that are used to prove the main result. For vector spaces X, Y, we denote by $L(X,Y)$ the set of all linear operators from X to Y. For an operator $T \in L(X,Y)$, $N(T)$ denotes the null space of T and $R(T)$ denotes the range of T. Thus $N(T) := \{x \in X : T(x) = 0\}$ and $R(T) := \{T(x) : x \in X\}$.

As usual, $L(X,X)$ will be denoted by $L(X)$. A map $P \in L(X)$ is called a *projection* if $P^2 = P$. Let $P \in L(X)$ and $Q \in L(Y)$ be projections. The restriction of QTP to $R(P)$ can be viewed as a linear operator from $R(P)$ to $R(Q)$. This is called a *section of T by P and Q* and will be denoted by $T_{P,Q}$. It is called a *finite section*, if $R(P)$ and $R(Q)$ are finite dimensional. When T is invertible, the section $T^{-1}_{I_Y-Q,I_X-P}$ of T^{-1} is called the *complementary section* of $T_{P,Q}$. With this terminology, our Null Space Theorem can be stated in the following very simple form:

There is a linear bijection between the null spaces of the complementary sections of T and T^{-1} (Theorem 2.1).

Its proof is also very simple. It is given in the next section. When X and Y are finite dimensional, T is represented by a matrix and complementary submatrices correspond to complementary sections (see [7]). Thus there is a linear bijection between the null spaces of the complementary submatrices of T and T^{-1}. Hence they have the same nullity. This is the Nullity Theorem (Theorem 1.1).

The authors of [7] have discussed several applications of the Nullity Theorem. For example, if T is an invertible tridiagonal matrix, then every submatrix of T^{-1} that lies on or above the main diagonal or on and below the main diagonal is of rank ≤ 1. However, proofs of these applications involve the famous Rank–Nullity Theorem (called the "Fundamental Theorem of Linear Algebra" in [6]) apart from the Nullity Theorem. Hence a straightforward imitation of these proofs to infinite dimensional case may or may not work, though the results may very well be true. Such an approach may work when the sections can be viewed as operators on finite dimensional spaces. In general, we need a different approach. This is attempted in the third section. We prove that if a tridiagonal operator on a Banach space with a Schauder basis is invertible, then certain sections of T^{-1} are of rank ≤ 1 (Theorem 3.1). This is followed by some illustrative examples and remarks about possible extensions.

2. Main result

Theorem 2.1 *(The Null Space Theorem). Let X, Y be vector spaces, $P \in L(X)$, $Q \in L(Y)$ be projections and $T \in L(X, Y)$ be invertible. Then there is a linear bijection between the null space of the section $T_{P,Q}$ of T and the null space of its complementary section $T^{-1}_{I_Y - Q, I_X - P}$ of T^{-1}.*

Proof. Let $x \in N(T_{P,Q})$. This means that $x \in R(P)$ so that $P(x) = x$ and $QTP(x) = 0$. Hence $QT(x) = 0$, that is, $(I_Y - Q)T(x) = T(x)$. Thus $T(x) \in R(I_Y - Q)$. Also, $(I_X - P)T^{-1}(I_Y - Q)T(x) = (I_X - P)T^{-1}T(x) = (I_X - P)(x) = 0$. Hence $T(x) \in N((I_X - P)T^{-1}(I_Y - Q))$ This means $T(x) \in N(T^{-1}_{I_Y - Q, I_X - P})$. This shows that the restriction of T to $N(T_{P,Q})$ maps $N(T_{P,Q})$ into $N((I_X - P)T^{-1}(I_Y - Q))$. Since T is invertible, this map is already injective. It only remains to show that it is onto. For this let $y \in N(T^{-1}_{I_Y - Q, I_X - P})$. This means $y \in R(I_Y - Q)$ and $(I_X - P)T^{-1}(I_Y - Q)(y) = 0$. We shall show that $T^{-1}(y) \in N(T_{P,Q})$. Since $y \in R(I_Y - Q)$, we have $(I_Y - Q)(y) = y$. Thus $Q(y) = 0$. Next, $0 = (I_X - P)T^{-1}(I_Y - Q)(y) = (I_X - P)T^{-1}(y)$. This implies that $PT^{-1}(y) = T^{-1}(y)$, that is, $T^{-1}(y) \in R(P)$. Also $QTPT^{-1}(y) = QTT^{-1}(y) = Q(y) = 0$. Thus $T^{-1}(y) \in N(QTP)$. Hence $T^{-1}(y) \in N(T_{P,Q})$. $\square$

Remark 2.2. As pointed out in the Introduction, this Null Space Theorem implies the Nullity Theorem (Theorem 1.1).

3. Ranks of submatirces

While considering infinite matrices, the products involve infinite sums, leading naturally to the questions of convergence. Hence it is natural to consider these questions in the setting of a Banach space X with a Schauder basis $A = \{a_1, a_2, \ldots\}$. We refer to [4] and [5] for elementary concepts in Functional Analysis.

Theorem 3.1. *Let X be a Banach space with a Schauder basis $A = \{a_1, a_2, \ldots\}$. Let T be a bounded (continuous) linear operator on X. Suppose the matrix of T with respect to A is tridiagonal. If T is invertible, then every submatrix of the matrix of T^{-1} with respect to A that lies on or above the main diagonal (or on or below the main diagonal) is of rank ≤ 1.*

Proof. Let M be the matrix of T with respect to A. Then M is infinite matrix of the form

$$M = \begin{bmatrix} \delta_1 & \alpha_1 & 0 & 0 & . \\ \beta_2 & \delta_2 & \alpha_2 & 0 & . \\ 0 & \beta_3 & \delta_3 & \alpha_3 & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

Then the matrix of T^{-1} with respect to A is M^{-1}. Let $M^{-1} = [\gamma_{i,j}]$. Let C_j denote the j-th column of M^{-1} and R_i denote the i-th row of M^{-1}. Thus

$$M^{-1} = [\, C_1 \quad C_2 \quad . \quad . \quad . \,] = \begin{bmatrix} R_1 \\ R_2 \\ . \\ . \\ . \end{bmatrix}$$

Further for a fixed natural number k, let C_j^k denote the column vector obtained by deleting the first $k-1$ entries from C_j. Thus

$$C_j^k = \begin{bmatrix} \gamma_{k,j} \\ \gamma_{k+1,j} \\ . \\ . \\ . \end{bmatrix}$$

Similarly, let R_i^k denote the row vector obtained by deleting first $k-1$ entries from R_i. Next let P_k denote the submatrix of M^{-1} given by

$$P_k = [\gamma_{i,j}, i \geq k, 1 \leq j \leq k] = \begin{bmatrix} C_1^k & C_2^k & . & . & C_k^k \end{bmatrix}$$

Similarly, let Q_k denote the submatrix of M^{-1} given by

$$Q_k = [\gamma_{i,j}, 1 \le i \le k, j \ge k] = \begin{bmatrix} R_1^k \\ R_2^k \\ . \\ . \\ R_k^k \end{bmatrix}$$

Note that every submatrix of M^{-1} that lies on or above the main diagonal is a submatrix of Q_k for some k and every submatrix of M^{-1} that lies on or below the main diagonal is a submatrix of P_k for some k. Thus it is sufficient to show that P_k and Q_k are of rank ≤ 1 for each k.

We shall give two proofs of this assertion.

First proof:

The assertion is evident for $k = 1$.

Now consider the equation $M^{-1}M = I$, that is,

$$[\,C_1 \quad C_2 \quad . \quad . \quad .\,] \begin{bmatrix} \delta_1 & \alpha_1 & 0 & 0 & . \\ \beta_2 & \delta_2 & \alpha_2 & 0 & . \\ 0 & \beta_3 & \delta_3 & \alpha_3 & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{bmatrix} = [\,e_1 \quad e_2 \quad . \quad . \quad .\,]$$

where, as usual, e_j denotes the column matrix whose j-th entry is 1 and all other entries are 0.

Equating the first columns on both sides of the above equation, we get

$$\delta_1 C_1 + \beta_2 C_2 = e_1$$

This, in particular, implies that at least one of δ_1, β_2 is not zero.

Deleting the first entries from all the column vectors in the above equation, we get

$$\delta_1 C_1^2 + \beta_2 C_2^2 = e_1^2 = 0$$

This shows that $\{C_1^2, C_2^2\}$ is a linearly dependent set, that is the matrix

$$P_2 = \begin{bmatrix} C_1^2 & C_2^2 \end{bmatrix}$$

is of rank ≤ 1.

Next we equate the second column on both sides of the equation. Then

$$\alpha_1 C_1 + \delta_2 C_2 + \beta_3 C_3 = e_2$$

Hence one of α_1, δ_2, β_3 is not zero.

Now deleting the first two entries from all the vectors appearing in this equation, we get

$$\alpha_1 C_1^3 + \delta_2 C_2^3 + \beta_3 C_3^3 = e_2^3 = 0$$

Since $\{C_1^2, C_2^2\}$ is a linearly dependent set, one of the vectors, say C_2^2 is a scalar multiple of the other, that is, C_1^2. This implies that C_2^3 is a scalar multiple of C_1^3. Now the above equation shows that C_3^3 is also a scalar multiple of C_1^3. Hence

$$P_3 = \begin{bmatrix} C_1^3 & C_2^3 & C_3^3 \end{bmatrix}$$

is of rank ≤ 1.

Proceeding in this way (more precisely, by Mathematical Induction) we can show that P_k is of rank ≤ 1 for each k.

Following essentially the same technique, equating the rows of both sides of the equation $MM^{-1} = I$, we can show that Q_k is of rank ≤ 1 for each k.

This completes the first proof.

Second proof:

Recall that since $A = \{a_1, a_2, \dots\}$ is a Schauder basis of X, every $x \in X$ can be expressed uniquely as $x = \sum_{j=1}^{\infty} \alpha_j a_j$ for some scalars α_j. Let X_n denote the linear span of $A_n := \{a_1, a_2, \dots, a_n\}$ and define a map $\pi_n : X \to X$ by $\pi_n(x) = \sum_{j=1}^{n} \alpha_j a_j$. Then π_n is a projection with $R(\pi_n) = X_n$. Also note that for each k, P_k as defined above is the matrix of the section $T^{-1}_{\pi_k, I-\pi_{k-1}}$ of the operator T^{-1}. As noted earlier, this can be viewed as an operator on $R(\pi_k) = X_k$. By the Null Space Theorem (Theorem 2.1), there is a linear bijection between the null space of this section and its complementary section, that is, the section $T_{\pi_{k-1}, I-\pi_k}$ of the operator T. This can be viewed as an operator on $R(\pi_{k-1}) = X_{k-1}$. It can be seen (in many ways) that this is in fact the zero operator on X_{k-1}. (The matrix of this section is the submatrix of M obtained by choosing the first $k-1$ columns and not choosing the first k rows. This is a zero matrix because M is tridiagonal.) Thus the null space of the section $T_{\pi_{k-1}, I-\pi_k}$ coincides with X_{k-1}. Hence the null space of the complementary section $T^{-1}_{\pi_k, I-\pi_{k-1}}$ is also of dimension $k-1$. This implies that its rank is 1 as it is an operator on X_k.

Thus P_k is of rank 1 for each k.

In a similar way, we can show that Q_k is of rank 1 for each k.

This completes the second proof. $\square$

Example 3.2. Let ℓ^2 denote the Hilbert space of square summable sequences and let $E = \{e_1, e_2, \dots\}$ be the orthonormal basis, where as usual e_j denotes the sequence whose j-th entry is 1 and all other entries are 0. Let R denote the Right Shift operator given by

$$R(x) = \big(0, x(1), x(2), \dots\big), \quad x \in \ell^2.$$

Consider a complex number c with $|c| < 1$ and let $T = I - cR$. Then the matrix of T with respect to the orthonormal basis E is tridiagonal and is given by

$$\begin{bmatrix} 1 & 0 & 0 & . & . \\ -c & 1 & 0 & 0 & . \\ 0 & -c & 1 & 0 & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

It can be easily checked that T is invertible and

$$T^{-1} = \sum_{j=0}^{\infty} c^j R^j$$

Thus the matrix of T^{-1} with respect to the orthonormal basis E is given by

$$\begin{bmatrix} 1 & 0 & 0 & . & . \\ c & 1 & 0 & 0 & . \\ c^2 & c & 1 & 0 & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

It is easily seen that every submatrix of the above matrix of T^{-1} that lies on or above (or on or below) the main diagonal is of rank 0 or 1.

Remark 3.3. In view of the above Theorem, T^{-1} or equivalently, $M^{-1} = [\gamma_{i,j}]$ can be described completely by using four sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ as follows: $\gamma_{i,j} = a_i b_j$ for $j \geq i$ and $\gamma_{i,j} = c_i d_j$ for $j \leq i$. Also, since for $i = j$, $\gamma_{i,i} = a_i b_i = c_i d_i$, these are essentially only three sequences. This should be expected as the tridiagonal operator T (matrix M) is completely described by three sequences, namely, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$. This can be useful in devising fast methods of computing T^{-1}. (See the Introduction of [7].)

Remark 3.4. It is also easy to see that the above proof can be easily modified in an obvious manner to a natural generalization that allows the matrix M of T to have a wider band. Suppose $M = [m_{i,j}]$ is such that $m_{i,j} = 0$ for $|i - j| > p$. (Thus $p = 1$ corresponds to tridiagonal operator.) Then using the same method, we can prove the following: every submatrix of the matrix of M^{-1} that lies above the pth subdiagonal or below the pth superdiagonal is of rank $\leq p$.

A careful look at the proof of Theorem 3.1 in fact shows that we have actually proved a more general result.

Theorem 3.5. *Let X be a Banach space with a Schauder basis $A = \{a_1, a_2, \ldots\}$. Let T be a bounded (continuous) linear operator on X. Suppose the matrix of T with respect to A*

is tridiagonal. If T has a bounded left inverse S, then every submatrix of the matrix of S with respect to A that lies on or below the main diagonal is of rank ≤ 1. Similarly, if T has a bounded right inverse U, then every submatrix of the matrix of U with respect to A that lies on or above the main diagonal is of rank ≤ 1.

Remark 3.6. As a simple example of the above Theorem 3.5, we may again consider the right shift operator R on ℓ^2 discussed in Example 3.2. Let L denote the Left Shift operator given by

$$L(x) = \big(x(2), x(3), \ldots\big), \quad x \in \ell^2.$$

Then L is a left inverse of R. Clearly, every submatrix of the matrix of L with respect to A that lies on or below the main diagonal is of rank ≤ 1.

Remark 3.7. Since left (right) inverse is one among the family of generalized inverses, Theorem 3.5 also raises an obvious question: Is there an analogue of Theorem 3.5 for other generalized inverses, in particular for Moore–Penrose pseudo-inverse? Such results are known for matrices. (See [1]) Information on generalized inverses of various types can be found in [3]

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