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APPROXIMATION OF MOORE-PENROSE INVERSE OF A CLOSED OPERATOR BY A SEQUENCE OF FINITE RANK OUTER INVERSES

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Abstract

Let T be a densely defined closed linear operator between complex Hilbert spaces H_1 and H_2 with domain $D(T) \subseteq H_1$ and separable range $R(T)$. In this note we approximate the Moore- Penrose inverse $T^\dagger$ of T by its finite rank bounded outer inverses. We also illustrate this method with an example.

1 Introduction

Suppose T is a densely defined closed linear operator between Hilbert spaces H_1 and H_2 with domain $D(T) \subseteq H_1$ and has a separable range. Let $T^\dagger$ denote the Moore-Penrose inverse of T. In this note we prove the following result:

For each $n \in \mathbb{N}$, there exists a bounded finite rank outer inverse $T_n^\#$ of T such that

$$T^\dagger y = \lim_{n\to\infty} T_n^\# y \quad \text{for all} \quad y \in D(T^\dagger).$$

Such methods were studied by Huang et al., [9] for the case of bounded operators with separable range. Earlier these results were proved for the bounded operators with closed and separable range by J. Ma and Z. Ma [15]. The proofs mentioned in these two articles are not applicable when the operators under consideration are unbounded. We overcome this difficulty by constructing new operators. The important point here is to note that a large number of the operators which arise naturally in applications e.g Mathematical Physics, Quantum Mechanics and Partial differential Equations are all unbounded (See [18, 19] for more details). In fact, many of these unbounded operators have compact inverse. To solve operator equations involving such unbounded operators it is necessary to generalize the existing results to the case of unbounded operators.

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The theory of generalized inverses has many applications in diverse mathematical fields like Optimization [8], Statistics, Economics, Games, Programming and Networks, Science and Engineering [1].

In this note we have made an attempt to generalize the existing results to the case of unbounded operators. A point worth noting is that while dealing with unbounded operators one has to be more careful with the domains of the operators as those are proper subspaces of the whole space. Hence, in most cases the techniques of the bounded operators do not work. The same is true for the Moore-Penrose inverse if the operator under consideration does not have closed range. This paper is a sequel to an earlier paper [10], in which we have discussed projection methods for the inverse of an unbounded operator.

The paper is organized in the following manner: In the second section we set up notations and state some of the definitions and results which will be frequently used throughout the remaining part of the paper. The third section contains the main results and an example to illustrate these results.

2 Notations and Basic results

Throughout the paper we consider the complex Hilbert spaces which will be denoted by H, H_1, H_2 etc . The inner product and the induced norm are denoted respectively by $\langle , \rangle$ and $||.||$. If $T : H_1 \to H_2$ is a linear operator with domain $D(T) \subseteq H_1$, then it is denoted by $T \in \mathcal{L}(H_1, H_2)$. The null space and range space of T are denoted by $N(T)$ and $R(T)$ respectively.

The graph of $T \in \mathcal{L}(H_1, H_2)$ is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$. If $G(T)$ is closed, then T is called a **closed operator**. The set of all closed operators is denoted by $\mathcal{C}(H_1, H_2)$. By the closed graph Theorem [6, Page 281], an everywhere defined closed operator is bounded. The set of all bounded operators is denoted by $\mathcal{B}(H_1, H_2)$. If $H_1 = H_2 = H$, then $\mathcal{B}(H_1, H_2)$ and $C(H_1, H_2)$ are denoted by $\mathcal{B}(H)$ and $\mathcal{C}(H)$ respectively.

If S and T are two linear operators such that $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$, then T is called a **restriction** of S and S is called an **extension** of T. We denote this fact by $T \subseteq S$.

If M is a closed subspace of a Hilbert space H, then P_M is the orthogonal projection onto M and $M^\perp$ is the orthogonal complement of M in H.

For closed subspaces M_1 and M_2 of H, the direct sum and the orthogonal direct sum are denoted by $M_1 \oplus M_2$ and $M_1 \oplus^\perp M_2$ respectively.

Definition 2.1. [1, Definition 1.12, Page 13] Let $T \in \mathcal{C}(H_1, H_2)$. If there exists an operator $T^\# \in \mathcal{L}(H_2, H_1)$ such that $T^\# T T^\# = T^\#$, then $T^\#$ is called an **outer inverse** of T (This is called $\{2\}$ **inverse** in [4]).

Definition 2.2. [4] Let $T \in \mathcal{L}(H_1, H_2)$. If $\overline{D(T)} = H_1$, then T is called **densely defined**. The subspace $C(T) := D(T) \cap N(T)^\perp$ is called the **carrier** of T.

Note 2.3. If $T \in \mathcal{C}(H_1, H_2)$, then $D(T) = N(T) \oplus^\perp C(T)$ [4, page 340].

Definition 2.4. [Moore-Penrose Inverse] [4] Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique densely defined operator $T^\dagger \in \mathcal{C}(H_2, H_1)$ with domain $D(T^\dagger) = R(T) \oplus^\perp R(T)^\perp$ and has the following properties;

1. $TT^\dagger y = P_{\overline{R(T)}}\, y,$ for all $y \in D(T^\dagger)$.
2. $T^\dagger T x = P_{N(T)^\perp}\, x,$ for all $x \in D(T)$.
3. $N(T^\dagger) = R(T)^\perp$.

This operator $T^\dagger$ is called the **Moore-Penrose inverse** of T.
The following property of $T^\dagger$ is also well known. For every $y \in D(T^\dagger)$, let

$$L(y) := \{x \in D(T) : ||Tx - y|| \leq ||Tu - y|| \ \forall\ u \in D(T)\}.$$

Here any $u \in L(y)$ is called a **least square solution** of the operator equation $Tx = y$. The vector $x = T^\dagger y \in L(y)$ and satisfies , $||T^\dagger y|| \leq ||x|| \ \forall\ x \in L(y)$ and is called the **least square solution of minimal norm**. A different treatment of $T^\dagger$ is described in [4, Pages 336, 339, 341], where the authors call it "**the Maximal Tseng generalized Inverse**".

Theorem 2.5. *[3, 5] Let $\{H_k\},\ k = 1, 2, 3, \ldots$ be closed subspaces of H and let $P_k = P_{H_k}$. Suppose $\{P_k\}$ is a monotone($H_k \subseteq H_{K+1}$ or $H_{k+1} \subseteq H_k$) sequence of orthogonal projections. Then the strong limit $P = \lim_{k\to\infty} P_{H_k}$ exists and P is the projection onto $\cap_k H_k$ in case P_k is non-increasing and onto $\overline{\cup_k H_k}$ if $\{P_k\}$ is non-decreasing.*

Proposition 2.6. *[4] Let $T \in \mathcal{C}(H_1, H_2)$ be a densely defined operator. Then*

1. *$N(T) = R(T^*)^\perp$*
2. *$N(T^*) = R(T)^\perp$*
3. *$N(T^*T) = N(T)$ and*
4. *$\overline{R(T^*T)} = \overline{R(T^*)}$.*

Proposition 2.7. *[7, 17] Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

1. *$(I + T^*T)^{-1} \in \mathcal{B}(H_1)$, $(I + TT^*)^{-1} \in \mathcal{B}(H_2)$.*
2. *$(I + TT^*)^{-1}T \subseteq T(I + T^*T)^{-1}$ and $||T(I + T^*T)^{-1}|| \leq 1$*
3. *$(I + T^*T)^{-1}T^* \subseteq T^*(I + TT^*)^{-1}$ and $||T^*(I + TT^*)^{-1}|| \leq 1$.*

3 Main Results

In this section, first we prove a lemma which is helpful in proving the main theorem.

Lemma 3.1. *Let* $T \in \mathcal{C}(H_1, H_2)$ *be densely defined. Let* $Y_n \subseteq R(T)$ *be such that*

(a) $Y_n \subseteq Y_{n+1}$ *for each* $n \in \mathbb{N}$

(b) $\dim Y_n = n$

(c) $\overline{\cup_{n=1}^{\infty} Y_n} = \overline{R(T)}$.

Let $Z_n := (I + TT^*)^{-1} Y_n$ *and* $X_n := T^* Z_n = T^*(I + TT^*)^{-1} Y_n$*. Then*

1. $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots \subseteq \overline{R(T^*)} = N(T)^{\perp}$, $\dim X_n = n$ *and*
2. $\overline{\cup_{n=1}^{\infty} Z_n} = \overline{R(T)}$
3. $\overline{\cup_{n=1}^{\infty} X_n} = \overline{R(T^*)}$
4. $\overline{\cup_{n=1}^{\infty} T X_n} = \overline{R(T)}$.

Proof. By the definition of X_n, $X_n \subseteq C(T) \subseteq N(T)^{\perp} = R(T^*)$ for all n and $X_n \subseteq X_{n+1}$. Since the operator $T^*(I + TT^*)^{-1}|_{\overline{R(T)}}$ is injective $\dim X_n = n = \dim Y_n$.

For a proof of (2), we make use of the following observation:

$$\overline{(I + TT^*)^{-1}(\overline{R(T)})} = \overline{R(T)}.$$

It can be proved easily that $(I + TT^*)^{-1}(N(TT^*)) = N(TT^*)$. By the Projection Theorem [13, 21.1, Page 420], $H_2 = N(TT^*) \oplus^{\perp} N(TT^*)^{\perp}$. That is $H_2 = N(TT^*) \oplus^{\perp} \overline{R(TT^*)}$. But

$$(I + TT^*)^{-1}(H_2) = D(TT^*) = N(TT^*) \oplus^{\perp} C(TT^*).$$

Hence

$$\begin{aligned} (I + TT^*)^{-1} H_2 &= (I + TT^*)^{-1}(N(TT^*) \oplus^{\perp} \overline{R(TT^*)}) \\ &= N(TT^*) \oplus^{\perp} (I + TT^*)^{-1}(\overline{R(TT^*)}). \end{aligned}$$

From this we can conclude that $(I+TT^*)^{-1}(\overline{R(TT^*)}) = C(TT^*)$ and as $\overline{C(TT^*)} = N(TT^*)^{\perp}$, we have $\overline{(I + TT^*)^{-1}(\overline{R(TT^*)})} = \overline{R(TT^*)}$. Hence $\overline{(I + TT^*)^{-1}(\overline{R(T)})} = \overline{R(T)}$, by Proposition (2.6). Thus

$$\begin{aligned} \overline{R(T)} = \overline{(I + TT^*)^{-1}(\overline{R(T)})} &= \overline{(I + TT^*)^{-1}(\overline{\cup_{n=1}^{\infty} Y_n})} \\ &= \overline{\cup_{n=1}^{\infty} (I + TT^*)^{-1} Y_n} \\ &= \overline{\cup_{n=1}^{\infty} Z_n}. \end{aligned}$$

This proves (2).

It is clear that $\overline{\cup_{n=1}^{\infty} X_n} \subseteq \overline{R(T^*)} = N(T)^{\perp}$.

Suppose $\overline{\cup_{n=1}^{\infty} X_n} \subsetneq N(T)^{\perp}$. Then there exists a $0 \neq z_0 \in N(T)^{\perp}$ such that $z_0 \in (\overline{\cup_{n=1}^{\infty} X_n})^{\perp}$. That is

$$\langle z_0, T^*(I+TT^*)^{-1}y\rangle = 0 \,\text{for all } y \in R(T)$$

By the continuity of $T^*(I+TT^*)^{-1}$, this holds for all $y \in \overline{R(T)}$.

We claim that this holds for all $y \in H_2$. Let $y \in H_2$. Then $y = u + v$ for some $u \in \overline{R(T)}$ and $v \in R(T)^{\perp} = N(T^*) \subseteq D(T^*)$. Hence by Proposition 2.7, $T^*(I+TT^*)^{-1}v = (I+T^*T)^{-1}T^*v = 0$. Hence

$$\langle z_0, T^*(I+TT^*)^{-1}y\rangle = \langle z_0, T^*(I+TT^*)^{-1}u\rangle = 0.$$

This proves the claim.

Next, since $\overline{C(T)} = N(T)^{\perp}$[12], there exists a sequence $\{z_n\} \subseteq C(T)$ such that $z_n \to z_0$. Hence for all $y \in H_2$,

$$\begin{aligned} 0 = \langle z_0, T^*(I+TT^*)^{-1}y\rangle &= \lim_{n\to\infty} \langle z_n, T^*(I+TT^*)^{-1}y\rangle \\ &= \lim_{n\to\infty} \langle Tz_n, (I+TT^*)^{-1}y\rangle \\ &= \lim_{n\to\infty} \langle (I+TT^*)^{-1}Tz_n, y\rangle \\ &= \lim_{n\to\infty} \langle T(I+T^*T)^{-1}z_n, y\rangle. \end{aligned}$$

This shows that $T(I+T^*T)^{-1}z_n \xrightarrow{w} 0$ (weakly), but since $T(I+T^*T)^{-1}$ is bounded, we have $T(I+T^*T)^{-1}z_0 = 0$. That is $(I+T^*T)^{-1}z_0 \in N(T)$. Let $y = (I+T^*T)^{-1}z_0$. Then $Ty = 0$. Hence $z_0 = (I+T^*T)y = y \in N(T)$. Thus $z_0 \in N(T) \cap N(T)^{\perp} = \{0\}$. Hence $z_0 = 0$, a contradiction to our assumption. This proves (3).

Using a similar argument we can prove (4). □

Remark 3.2. We may note that Lemma 3.1 implies that if $R(T)$ is separable, then $R(T^*)$ is separable. This generalizes an analogous result for bounded operators proved in [2, Page 362].

Theorem 3.3 (Compare Theorem 2.1 of [9])**.** *Let $T \in \mathcal{C}(H_1, H_2)$ be a densely defined operator with separable range $R(T)$. Then for each $n \in \mathbb{N}$, there exists a bounded outer inverse $T_n^{\#}$ of T of rank n such that*

$$T^{\dagger}y = \lim_{n\to\infty} T_n^{\#}y \quad \textit{for all} \quad y \in D(T^{\dagger}).$$

Proof. Assume that $R(T)$ is infinite dimensional. Since $R(T)$ is separable, we can find a sequence of subspaces of Y_n of $\overline{R(T)}$ with the following properties:

1. $Y_n \subseteq Y_{n+1}$ and $\dim Y_n = n$ for all $n \in \mathbb{N}$.
2. $\overline{\cup_{n=1}^{\infty} Y_n} = \overline{R(T)}$.

(For example if $\{\phi_1, \phi_2, \ldots,\}$ is an orthonormal set that spans $R(T)$, then define $Y_n := \text{span}(\{\phi_1, \phi_2, \ldots, \phi_n\})$).

Let Z_n and X_n be as in Lemma 3.1. Then $Z_n \subseteq Z_{n+1}$ and $\dim Z_n = n$. Similar results hold for X_n.

Let $P_n : H_2 \to H_2$ and $Q_n : H_1 \to H_1$ be sequences of orthogonal projections with $R(P_n) = Z_n$ and $R(Q_n) = X_n$. Let $T_n := P_n T$. Here $D(T_n) = D(T)$ and $T_n x \to Tx$ for all $x \in D(T)$.

Next we claim that $R(T_n) = R(P_n) = Z_n$. It is clear that $R(T_n) \subseteq R(P_n) = Z_n$. To show the other way inclusion, it is enough to show $N(T_n^*) \subseteq N(P_n)$. Now let $z \in N(T_n^*)$. Then $T^* P_n z = 0$. Hence $P_n z \in N(T^*) = R(T)^\perp$. But, $P_n z \in \overline{R(T)}$. Hence $P_n z = 0$. Thus $z \in N(P_n)$. Note that $T_n^* = T^* P_n = T^*|_{R(P_n)} = T^*|_{Z_n}$. Hence $R(T_n^*) = T^* Z_n = X_n = N(T_n)^\perp$. That is $N(T_n) = X_n^\perp$. $R(T_n)^\perp = N(T_n *) = Z_n^\perp$. That is $R(T_n) = Z_n$. So $T_n|_{X_n} : X_n \to Z_n$ is a bijective operator. Hence $\dim X_n = \dim Z_n = n$.

Construction of outer inverses: Define $T_n^\# : H_2 \to H_1$ by

$$T_n^\# y := \begin{cases} (T_n|_{X_n})^{-1} y, & \text{if } y \in Z_n, \\ 0, & \text{if } y \in Z_n^\perp. \end{cases}$$

Here $T_n^\# = T_n^\dagger$ and $T_n^\#$ is bounded since $R(T_n)$ is closed. It is also true that $T_n^\#$ is an outer inverse of T_n. Here $N(T_n^\#) = Z_n^\perp$ and $R(T_n^\#) = X_n$.

Next we claim that $T_n^\#$ is also an outer inverse of T. For this we make use of the following observation: $T_n^\# y = T_n P_n y$, for all $y \in H_2$. To see this let $y \in H_2$. Then $y = u + v$ for some $u \in Z_n$ and $v \in Z_n^\perp$.

Hence

$$\begin{aligned} T_n^\# y = T_n^\#(u+v) &= T_n^\# u \qquad (\because T_n^\#(v) = 0, \quad \text{because}\, v \in Z_n^\perp) \\ &= T_n^\# P_n y. \end{aligned}$$

Since $T_n^\#$ is an outer inverse of T_n,

$$\begin{aligned} T_n^\# T T_n^\# y = T_n^\# P_n T T_n^\# y &= T_n^\# T_n T_n^\# y \\ &= T_n^\# y. \end{aligned}$$

Our next aim is to show that $\lim_{n\to\infty} T_n^\# y$ exists and equals $T^\dagger y$ for all $y \in D(T^\dagger)$. Let $y \in D(T^\dagger)$. Then $T^\dagger y \in C(T)$. Since $Q_n x \to x$ for all $x \in C(T) \subseteq N(T)^\perp = \overline{R(T^*)}$, it is clear that $Q_n T^\dagger y \to T^\dagger y$. Next we show that $Q_n T^\dagger y = T_n^\# y$, for all $y \in D(T^\dagger)$.

From the facts $Q_nT^\dagger y \in X_n, (Q_n - I)T^\dagger y \in N(T_n)$ and Theorem 2.5,

$$\begin{aligned} Q_nT^\dagger y = T_n^\# T_n Q_n T^\dagger y &= T_n^\# T_n Q_n T^\dagger y + T_n^\# P_n y - T^\# P_n y \\ &= T_n^\# (T_n Q_n T^\dagger y - P_n y) + T^\# P_n y \\ &= T_n^\# (T_n Q_n T^\dagger y - P_n T T^\dagger y) + T_n^\# P_n y \\ &= T_n^\# (T_n Q_n - P_n T) T^\dagger y + T_n^\# P_n y \\ &= T_n^\# T_n (Q_n - I) T^\dagger y + T_n^\# P_n y \\ &= T_n^\# P_n y \\ &= T_n^\# y. \end{aligned}$$

As $Q_nT^\dagger y \to T^\dagger y$ for all $y \in D(T^\dagger)$, and by the above argument $\lim_{n\to\infty} T_n^\# y$ exists and equals $T^\dagger y$. □

Theorem 3.4 (Compare Corollary 2.1 of [9])**.** *Let $T \in \mathcal{C}(H_1, H_2)$ be a densely defined operator. Then the following statements are equivalent:*

1. *$R(T)$ is closed.*
2. *$T^\dagger$ is bounded.*
3. *$D(T^\dagger) = H_2$.*
4. *0 is not an accumulation point of $\sigma(T^*T)$.*

   If, in addition $R(T)$ is separable and $T_n^\#$ are as in Theorem 3.3, then each of the above statements is also equivalent to each of the following;

5. *$\lim_{n\to\infty} T_n^\# y$ exists for all $y \in H_2$.*
6. *$T_n^\#$ is uniformly bounded.*

Proof. The equivalence of (1), (2) and (3) is well known and can be found in ([4]).
The equivalence of (1) and (4) is proved in ([14, Theorem 3.3]).
The equivalence of (3) and (5) follows from Theorem 3.3.
The implication (5) ⇒ (6) follows from the Uniform boundedness principle.
(6) ⇒ (5):
By Theorem 3.3, $\lim_{n\to\infty} T_n^\# y$ exists for every $y \in D(T^\dagger)$. Since $\overline{D(T^\dagger)} = H_2$ [4, Theorem 2 , Page 320], the conclusion follows by [16, Theorem 6.4, Page 220]. □

Remark 3.5. The authors in [9] proved that if $T \in \mathcal{B}(H_1, H_2)$ with a separable range, then for each $n \in \mathbb{N}$, there exists a bounded outer inverse $T_n^\#$ of T of rank n such that

$$D(T^\dagger) = \{y \in H_2 : \lim_{n\to\infty} T_n^\# y \quad \text{exists}\}$$

and

$$T^\dagger y = \lim_{n\to\infty} T_n^\# y \quad \text{for all} \quad y \in D(T^\dagger).$$

Using similar arguments as in Theorem 3.3, we can prove the following result.

Theorem 3.6. *Let $T \in \mathcal{C}(H_1, H_2)$ be a densely defined closed operator. If there exists a sequence of increasing orthogonal projections P_n on H_2 onto subspace of $\overline{R(T)}$ with the property that $P_n y \to P_{\overline{R(T)}} y$ for all $y \in H_2$ and $R(P_n T)$ is closed, then for each n, there exists an outer inverse $T_n^{\#}$ such that*

$$T^{\dagger} y = \lim_{n\to\infty} T_n^{\#} y \quad \text{for all} \quad y \in D(T^{\dagger}).$$

Example 3.7. Let $T : \ell^2 \to \ell^2$ be with

$$D(T) := \{(x_1, x_2, \dots) \in \ell^2 : (0, 2x_2, 0, 4x_4, \dots) \in \ell^2\}.$$

Define

$$T(x_1, x_2, \dots) = (0, 2x_2, 0, 4x_4, \dots) \text{ for all } (x_1, x_2, \dots) \in D(T).$$

It can be shown that $T = T^*$ and $R(T)$ is closed. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthogonal basis for ℓ^2. Here $R(T) = \overline{\text{span}(e_2, e_4, \dots, e_{2n}, \dots)}$. Let $Y_n := \text{span}\{e_2, e_4, \dots, e_{2n}\}$. Then $Y_n \subseteq Y_{n+1}$, $\dim(Y_n) = n$ and $\overline{\cup_{n=1}^{\infty} Y_n} = R(T)$. Since $T = T^*$, we have $I + TT^* = I + T^2$.

For any $x = (x_1, x_2, \dots) \in D(T^2)$,

$$(I + T^2)x = (x_1, 5x_2, x_3, 17x_4, \dots, x_{2n-1}, (1 + 4n^2)x_{2n}, \dots).$$

For any $y = (y_1, y_2, \dots) \in \ell^2$,

$$(I + T^2)^{-1} y = (y_1, \frac{y_2}{5}, y_3, \frac{4}{17} y_4, \dots, y_{2n-1}, \frac{y_{2n}}{1 + 4n^2}, \dots), \quad y = (y_1, y_2, \dots,) \in \ell^2.$$

In particular, $(I + T^2)^{-1}(e_{2n}) = \dfrac{e_{2n}}{1 + 4n^2}$. Hence $Z_n = (I + T^2)^{-1} Y_n = Y_n$. Also $X_n = T^* Z_n = Y_n$. Then $X_n = Y_n = Z_n$. Hence $P_n = Q_n$. That is $P_n x = x_2 e_2 + x_4 e_4 + \dots + x_{2n} e_{2n}$ for all $x = (x_1, x_2, \dots, x_n, \dots,) \in \ell^2$. $T_n = P_n T$. There fore $T_n x = 2x_2 e_2 + 4x_4 e_4 + \dots + 2n x_{2n} e_{2n}$. Hence

$$T_n^{\#}(y) = \begin{cases} \frac{y_2}{2} e_2 + \frac{y_4}{4} e_4 + \dots \frac{y_{2n}}{2n} e_{2n} & \text{if } y \in Y_n \\ 0 & \text{if } y \in Y_n^{\perp}. \end{cases}$$

Hence by Theorem 3.3, $D(T^{\dagger}) = \{y \in \ell^2 : \lim_{n\to\infty} T_n^{\#} \text{ exists}\} = \ell^2$ and

$$T^{\dagger} y = \lim_{n\to\infty} T_n^{\#} = \lim_{n\to\infty} \big((0, \frac{1}{2} y_2, 0, \frac{1}{4} y_4)\big) \text{ for all } y = (y_1, y_2, \dots) \in \ell^2.$$

Concluding remarks: For an arbitrary densely defined closed operator T, computing Z_n, X_n may be difficult. A procedure to compute $(I + TT^*)^{-1}$ is indicated in [7]. We hope to apply this procedure to some concrete densely defined closed operators in future.

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