Shift invariant spaces in $$L^2({\mathbb {R}},{\mathbb {C}}^m)$$ L 2 (R , C m) with m generators

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Shift invariant spaces in $L^2(\mathbb{R},\mathbb{C}^m)$ with m generators

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Abstract
The paper deals with sampling and reconstruction of vector valued functions in a shift invariant space with multiple generators. Unlike the case of a shift invariant space with multiple generators in $L^2(\mathbb{R})$, when the dimension of the vectors is the same as the number of generators, $\mathbb{Z}$ turns out to be a stable set of sampling. A sampling formula for reconstructing a function from its samples at integer points is derived and the problem of sampling on a perturbed set of integers is discussed. An illustration of sampling and reconstruction of a function in $L^2(\mathbb{R},\mathbb{R}^2)$ on a finite interval is given using Matlab.

Keywords Block Laurent operator · Reproducing kernel Hilbert space · Stable set of sampling · Vector valued Zak transform

Mathematics Subject Classification 42C15 · 94A20

1 Introduction

The theoretical aspects of non uniform sampling began to develop significantly from the mid twentieth century. Yet it is known to have its roots in the 1841 work of Cauchy (see [18]). Generally speaking, the sampling problem refers to finding out a function $f : \mathbb{R} \to \mathbb{C}$, from a countable number of samples of f given by, say, $f(x_k)$:

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$k \in \mathbb{I}$ where $\mathbb{I}$ is a countable index set. Stated in this manner, the problem is ill-posed, since there can be uncountable number of functions passing through a given countable set of points. Hence sampling problems have been studied in convenient spaces of functions, namely, shift invariant spaces.

The classical *Shannon sampling theorem* characterizes sampling and reconstruction in the shift invariant space of bandlimited functions generated by the sinc function. The sinc function has the property that its Fourier transform is compactly supported. Let $B_{[-\frac{1}{2},\frac{1}{2}]}$ denote the collection of functions f whose Fourier transform $\widehat{f}$ has support $[-\frac{1}{2},\frac{1}{2}]$. If $f \in L^2(\mathbb{R}) \cap B_{[-\frac{1}{2},\frac{1}{2}]}$ and $f(k)$ are its samples taken at the integer points $k \in \mathbb{Z}$, then the Shannon sampling theorem states that f can be reconstructed by the formula

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) sinc(x-k).$$

Paley and Weiner [22] extended this result to non-uniform samples instead of integers. They proved that if $X = \{x_k : k \in \mathbb{Z}\}$ is such that $|x_k - k| \leq \frac{1}{\pi^2}$, then any band limited function can be reconstructed from the samples $\{f(x_k) : k \in \mathbb{Z}\}$. Duffin and Eachus [8] improved this gap between x_k and k to 0.22. Later, Kadec in [14] proved that the maximum gap between x_k and k has to be less than $\frac{1}{4}$.

For a more general sampling set, the sampling condition for band limited functions is given in terms of the Beurling density $D(X)$ defined as follows:

$$D(X) = \lim_{r \to \infty} \inf_{y \in \mathbb{R}} \frac{\#[X \cup (y + [0,r])]}{r}.$$

In fact, if $D(X) > 1$, then any $f \in L^2(\mathbb{R}) \cap B_{[-\frac{1}{2},\frac{1}{2}]}$ can be reconstructed from its samples $\{f(x_k) : k \in X\}$ stably and uniquely. Conversely, if every $f \in L^2(\mathbb{R}) \cap B_{[-\frac{1}{2},\frac{1}{2}]}$ can be uniquely and stably reconstructed from its samples, then $D(X) \geq 1$ (see [16]). For a comprehensive survey of sampling theory we refer to the classical work of Butzer and Stens in [7]. It traces the theory of sampling from the work of de la Vaullee Poussin in 1908 (see [18]).

Non uniform sampling in shift invariant spaces gained further attention following the invention of orthogonal wavelet bases and multiresolution analysis by Meyer [20] and Mallat [19]. The problem of sampling and reconstruction is extensively studied in shift invariant spaces with single generators. See, for example, [2, 11, 17, 26].

However, the problem of stable set of sampling was not studied much in the case of a shift invariant space with multiple generators until recently. In [24], the authors address the problem of obtaining conditions under which $\mathbb{Z}$ can be a stable set of sampling for a shift invariant space with multiple generators. Surprisingly, it turns out that $\mathbb{Z}$ can never become a stable set of sampling when the number of generators is ≥ 2. In [23], the authors discuss the problem of obtaining conditions under which $\mathbb{Z}/m, m \in \mathbb{N}$ and $r\mathbb{Z}, r \in \mathbb{Q}^+$ become stable set of sampling. But, there are several interesting results concerning sampling and reconstruction in a shift invariant space with multiple generators obtained by several authors. Some problems in this direction are average sampling [5, 27, 30, 31] dynamic sampling [32], random sampling [10], stable reconstruction models with respect to different kinds of small perturbations [1, 17], robustness of the sampling procedure [3], sampling and reconstruction in reproducing kernel subspaces of $L^p(\mathbb{R}^d)$ [4, 21, 33], multiwavelet spaces [29], local reconstruction procedures [13, 25, 28] and so on.

The problem of interest in the present paper is to study the stable set of sampling and reconstruction of vector valued functions in a shift invariant space with multiple generators. Following an intuitive lead to the invertibility of block Laurent operators, we investigate the stable sets of sampling for $V(F)$, the shift invariant space of vector valued functions generated by integer translates of elements of F, where $F = \{\phi_1, \ldots, \phi_m\}, \phi_i \in L^2(\mathbb{R}, \mathbb{C}^m), i = 1, \ldots, m$. Unlike the case of shift invariant spaces with multiple generators in $L^2(\mathbb{R})$, it turns out that when the dimension of the vectors is the same as the number of generators, $\mathbb{Z}$ becomes a stable set of sampling. Firstly, we derive the condition for the set of translates $\{\tau_n\phi_i, n \in \mathbb{Z}, i = 1, \ldots, m\}$ to be a Riesz basis for $V(F)$. A Zak transform for the vector valued functions is defined and it is used to obtain an equivalent condition for a stable set of sampling for $V(F)$. In fact, a set of equivalent conditions for a countable indexed set X to be a stable set of sampling for $V(F)$ is derived.

As in [23, 26], a sampling formula for reconstructing a function $f \in V(F)$ from its samples at integer points is derived. Finally, the problem of sampling on a perturbed set of integers is also discussed. An illustration is given using Matlab for sampling and reconstruction of a function in $L^2(\mathbb{R}, \mathbb{R}^2)$ on a finite interval.

2 Definitions and background

In this section, we shall provide the necessary definitions and background for the remaining sections of the paper.

Definition 2.1 Let $L^2(\mathbb{R}, \mathbb{C}^m)$ denote the space of vector valued square integrable functions $f : \mathbb{R} \to \mathbb{C}^m$. The inner product on this space is given by

$$\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{C}^m)} = \int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}^m} dx = \int_{\mathbb{R}} g(x)^* \cdot f(x) dx = \int_{\mathbb{R}} \sum_{i=1}^{m} \overline{g_i(x)} f_i(x) dx.$$

Definition 2.2 A closed subspace M of $L^2(\mathbb{R}, \mathbb{C}^m)$ is called a shift invariant space if $\tau_n\phi \in M$ for all $n \in \mathbb{Z}$ and for all $\phi \in M$. Here τ_n is the translation operator defined by $\tau_n\phi(x) = \phi(x - n)$.

We shall work with the following shift invariant space, with m generators in $L^2(\mathbb{R}, \mathbb{C}^m)$. Let $F = \{\phi_1, \ldots\ldots, \phi_m\} \subset L^2(\mathbb{R}, \mathbb{C}^m)$. Define

$$V(F) = \overline{span\{\tau_n\phi_1, \ldots, \tau_n\phi_m; n \in \mathbb{Z}\}}.$$

Definition 2.3 Let $0 \neq \mathbb{H}$ denote a separable Hilbert space. A sequence of vectors $\{f_n, n \in \mathbb{Z}\}$ in $\mathbb{H}$ is called a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B||f||^2, \tag{1}$$

for all $f \in \mathbb{H}$. The numbers A,B are called the frame bounds.

Definition 2.4 ([6]) Let $\mathbb{H}_1, \ldots, \mathbb{H}_L$ be separable Hilbert spaces. Let $\mathbb{F}_i \subset \mathbb{H}_i, i = 1, \ldots, L$. Let $\pi_i : \mathbb{I} \to \mathbb{F}_i, i = 1, \ldots, L$ be indexing maps where $\mathbb{I}$ is a countable index set. Let $\mathbb{F} \subset \mathbb{H}_1 + \ldots + \mathbb{H}_L$. Then,

$$\begin{aligned} \mathbb{F} &= \mathbb{F}_1 + \ldots + \mathbb{F}_L \\ &= \{f_i^1 + \cdots + f_i^L, i \in \mathbb{I}\}, \quad f_i^k = \pi_k(i) \in \mathbb{F}_k. \end{aligned}$$

Then the set $\{\pi_1(i), \ldots, \pi_L(i) : i \in \mathbb{I}\}$ is called a *super frame*, if there exist constants $0 < A \leq B < \infty$ such that for each $h_k \in \mathbb{H}_k, 1 \leq k \leq L$,

$$A(||h_1||^2 + \cdots + ||h_L||^2) \leq \sum_{n\in\mathbb{I}} \left(\sum_{k=1}^{L} |\langle h_k, f_n^k \rangle|^2 \right) \leq B(||h_1||^2 + \cdots + ||h_L||^2). \tag{2}$$

Definition 2.5 A sequence of vectors $\{f_n, n \in \mathbb{Z}\}$ in a separable Hilbert space $\mathbb{H}$ is called a *Riesz basis*, if $\overline{span\{f_n\}} = \mathbb{H}$, and there exist constants $0 < c < C < \infty$ such that

$$c \sum_{n\in\mathbb{Z}} |d_n|^2 \leq || \sum_{n\in\mathbb{Z}} d_n f_n||^2 \leq C \sum_{n\in\mathbb{Z}} |d_n|^2,$$

for all $(d_n) \in \ell^2(\mathbb{Z})$.

Definition 2.6 For $F = \{\phi_1, \ldots, \phi_m\} \subset L^2(\mathbb{R}, \mathbb{C}^m)$, let $\widehat{\phi_j}$ denote the Fourier transform of ϕ_j. Then the Gramian matrix $G_{F(\xi)} = [G_{\phi_i,\phi_j}(\xi)]$ is defined as follows.

$$G_{\phi_i,\phi_j}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \widehat{\phi_i}(\xi + n), \widehat{\phi_j}(\xi + n) \right\rangle; i,j = 1, \ldots, m \quad \xi \in \mathbb{R}.$$

In particular, when $\{\phi_1, \ldots, \phi_m\} \subset L^2(\mathbb{R})$, it has been proved in [9] that $\{\tau_n \phi_i, n \in \mathbb{Z}, i \in \{1, \ldots., m\}\}$ is a Riesz basis for $V(F)$ iff there exist $\alpha > 0, \beta > 0$ such that

$$\alpha I \preceq G_{F(\xi)} \preceq \beta I \quad a.e. \, \xi \in \mathbb{R}. \tag{3}$$

We shall prove a similar theorem for functions in $L^2(\mathbb{R}, \mathbb{C}^m)$ in the next section.

Definition 2.7 A vector valued reproducing kernel Hilbert space, denoted by RKHS, is a Hilbert space $\mathbb{H}$ of functions $f : \mathbb{R} \to \mathbb{C}^m$, if there exist a function

$K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{m\times m}$, such that for every $c \in \mathbb{C}^m$ and $x \in \mathbb{R}$, $K(x, x')c$ belongs to $\mathbb{H}$. If we denote $K_x(y) = K(y, x)$ for a given x and $\forall y \in \mathbb{R}$, we have

$$\langle K_x c, f\rangle = \langle c, f(x)\rangle,$$

where the LHS is an inner product in $\mathbb{H}$. The function K is called the reproducing kernel of $\mathbb{H}$ and $K(x, x')$ is a positive semi definite matrix for each x, x'.

Definition 2.8 A bounded linear operator $L : l^2(\mathbb{Z})^m \longrightarrow l^2(\mathbb{Z})^m$ is a *block Laurent operator* iff its matrix has the form

$$\begin{bmatrix} \ddots & & & & \\ & A_0 & A_{-1} & A_{-2} & \\ & A_1 & A_0 & A_{-1} & \\ & A_2 & A_1 & A_0 & \\ & & & & \ddots \end{bmatrix}. \tag{4}$$

Here $l^2(\mathbb{Z})^m$ is a direct sum of m copies of $l^2(\mathbb{Z})$. Therefore, an operator L on $l^2(\mathbb{Z})^m$ can also be represented as an $m \times m$ matrix whose entries are operators acting on $l^2(\mathbb{Z})$. Thus,

$$L = \begin{bmatrix} L_{1,1} & \dots & L_{1,m} \\ \vdots & \vdots & \\ L_{m,1} & \dots & L_{m,m} \end{bmatrix} : l^2(\mathbb{Z})^m \longrightarrow l^2(\mathbb{Z})^m. \tag{5}$$

It can be shown that L is a Laurent operator on $l^2(\mathbb{Z})^m$ iff all the entries L_{rs} in the matrix representation (5) are Laurent operators on $l^2(\mathbb{Z})$. Also we have

$$L_{rs} = [A^{rs}_{i-j}]^{\infty}_{i,j=-\infty},$$

where A^{rs}_n is the (r, s)-th entry of the matrix A_n with respect to the standard basis of $\mathbb{C}^m$. Thus we have,

$$L_{rs}(\ldots, 0, 0, \boxed{1}, 0, 0, \ldots) = (A^{rs}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

If $z \in S^1$, where $S^1 = \{z \in \mathbb{C}, |z| = 1\}$, put

$$\Psi(z) = \begin{bmatrix} \Psi_{1,1}(z) & \dots & \Psi_{1,m}(z) \\ \vdots & \vdots & \vdots \\ \Psi_{m,1}(z) & \dots & \Psi_{m,m}(z) \end{bmatrix}, \tag{6}$$

where,

$$\Psi_{rs}(z) = \sum_{-\infty}^{\infty} z^n A_n^{rs}.$$

Ψ is called the defining function or symbol of the Laurent operator L.

Theorem 2.9 [12] *Let L be the block Laurent operator with symbol Φ. Then L is invertible iff there exists $\gamma > 0$ such that*

$$|det\Phi(z)| \geq \gamma \quad a.e. \, z \in S^1 \tag{7}$$

and in this case L^{-1} is the Laurent operator defined by $\Phi(.)^{-1}$.

3 The main results

Theorem 3.1 *Let $F = \{\phi_1, \ldots\ldots, \phi_m\} \subset L^2(\mathbb{R}, \mathbb{C}^m)$. Then $\{\tau_n\phi_i, \quad n \in \mathbb{Z}, i \in \{1, \ldots.., m\}\}$ is a Riesz Basis for $V(F)$ iff there exist $\alpha > 0, \beta > 0$ such that*

$$\alpha I \preceq G_{\mathbb{F}(\xi)} \preceq \beta I \quad a.e. \quad \xi \in \mathbb{R} \tag{8}$$

where,

$$G_{F(\xi)} = \begin{bmatrix} G_{\phi_1,\phi_1}(\xi) & \ldots & G_{\phi_1,\phi_m}(\xi) \\ \vdots & \vdots & \vdots \\ G_{\phi_m,\phi_1}(\xi) & \ldots & G_{\phi_m,\phi_m}(\xi) \end{bmatrix}. \tag{9}$$

Proof Suppose $\{\tau_n\phi_i, \quad n \in \mathbb{Z}, i \in \{1, \ldots.., m\}\}$ is a Riesz basis for $V(F)$. Then there exist constants $0 < c < C < \infty$ such that,

$$c \sum_{n\in\mathbb{Z}} ||d_n||^2 \leq || \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} d_l(n)\tau_n\phi_l||^2 \leq C \sum_{n\in\mathbb{Z}} ||d_n||^2,$$

for all $(d) = (d_1, \ldots, d_m) \in \ell^2(\mathbb{Z})^m$.

Now

$$\begin{aligned} || \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} d_l(n)\tau_n\phi_l||^2 &= \left\langle \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} d_l(n)\tau_n\phi_l, \sum_{r\in\mathbb{Z}} \sum_{k=1}^{m} d_k(r)\tau_r\phi_k \right\rangle \\ &= \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{k=1}^{m} \int_{-\infty}^{\infty} \overline{d_k(r)} d_l(n) \overline{\phi_k(t-r)} \cdot \phi_l(t-n) dt \\ &= \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{k=1}^{m} \overline{d_k(r)} d_l(n) \int_{-\infty}^{\infty} \overline{\phi_k(t)} \cdot \phi_l(t-(n-r)) dt \\ &= \sum_{s\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{k=1}^{m} \overline{d_k(r)} d_l(s+r) \int_{-\infty}^{\infty} \overline{\phi_k(t)} \cdot \phi_l(t-s) dt, \end{aligned}$$

by applying change of variables. Taking Fourier transform, we get $\| \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} d_l(n)\tau_n\phi_l\|^2$

$$
\begin{aligned}
&= \sum_{s\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{k=1}^{m} \overline{d_k(r)} d_l(s+r) \int_{-\infty}^{\infty} \overline{\widehat{\phi}_k(\omega)} \cdot \widehat{\phi}_l(\omega) e^{-j\omega s} d\omega \\
&= \sum_{l=1}^{m} \sum_{k=1}^{m} \int_{-\infty}^{\infty} \sum_{s\in\mathbb{Z}} d_l(s+r) e^{-j\omega(s+r)} \sum_{r\in\mathbb{Z}} \overline{d_k(r) e^{-j\omega r} \widehat{\phi}_k(\omega)} \cdot \widehat{\phi}_l(\omega) d\omega \\
&= \int_{-\infty}^{\infty} \sum_{l=1}^{m} \sum_{k=1}^{m} \overline{\widehat{d}_k(\omega)} \widehat{d}_l(\omega) \overline{\widehat{\phi}_k(\omega)} \cdot \widehat{\phi}_l(\omega) d\omega \\
&= \overline{\widehat{d}(\omega)} M \widehat{d}(\omega)
\end{aligned}
$$

where,

$$
\widehat{d}(\omega) = (\widehat{d}_1(\omega), \ldots, \widehat{d}_m(\omega))
$$

and

$$
M = \begin{bmatrix} \left\langle \widehat{\phi}_1(\omega), \widehat{\phi}_1(\omega) \right\rangle & \ldots & \left\langle \widehat{\phi}_m(\omega), \widehat{\phi}_1(\omega) \right\rangle \\ \vdots & \vdots & \vdots \\ \left\langle \widehat{\phi}_1(\omega), \widehat{\phi}_m(\omega) \right\rangle & \ldots & \left\langle \widehat{\phi}_m(\omega), \widehat{\phi}_m(\omega) \right\rangle \end{bmatrix}, \tag{10}
$$

which implies that $\alpha I \preceq G_{\mathbb{F}(\xi)} \preceq \beta I \quad a.e. \, \xi \in \mathbb{R}$. By retracing the steps, we arrive at the converse part. □

Now we shall show that $V(F)$ is a RKHS. We have $F = \{\phi_1, \ldots\ldots, \phi_m\} \subset L^2(\mathbb{R}, \mathbb{C}^m)$. We shall denote $\phi_i = (\phi_{i,1}, \ldots, \phi_{i,m}), \quad \forall i \in \{1, \ldots, m\}$ where each $\phi_{i,j} \in L^2(\mathbb{R}); i, j \in \{1, \ldots, m\}$.

Theorem 3.2 *The shift invariant space V(F) is a reproducing kernel Hilbert space.*

Proof Define $K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{m\times m}$ by

$$
K(x,y) = \begin{bmatrix} \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,1}(y), \tau_n \phi_{l,1}(x) \right\rangle & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & \ldots & 0 & \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,m}(y), \tau_n \phi_{l,m}(x) \right\rangle \end{bmatrix}.
$$

using the dual basis $\{\tau_n \tilde{\phi}_i, \quad n \in \mathbb{Z}, i \in \{1, \ldots., m\}\}$. Then for $c \in \mathbb{C}^m$,

$$
K(\cdot, x)c = (c_1 \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,1}(\cdot), \tau_n \phi_{l,1}(x) \right\rangle, \cdots, c_m \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,m}(\cdot), \tau_n \phi_{l,m}(x) \right\rangle)^T
$$

where,

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

Let $f = (f_1, \ldots, f_m)$. Now,

$$\begin{aligned} &\langle K(.,x)c, f\rangle \\ &= \left\langle \left(c_1 \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,1}(.), \tau_n \phi_{l,1}(x) \right\rangle, \ldots, \right.\right. \\ &\quad \left.\left. c_m \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle \tau_n \tilde{\phi}_{l,m}(.), \tau_n \phi_{l,m}(x) \right\rangle \right)^T, (f_1, \ldots, f_m)^T \right\rangle \\ &= \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \langle c, \tau_n \phi_l(x)\rangle \left\langle \tau_n \tilde{\phi}_l(.), f \right\rangle \\ &= \left\langle c, \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \overline{\left\langle \tau_n \tilde{\phi}_l(.), f \right\rangle} \tau_n \phi_l(x) \right\rangle \\ &= \left\langle c, \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \left\langle f, \tau_n \tilde{\phi}_l(.) \right\rangle \tau_n \phi_l(x) \right\rangle \\ &= \langle c, f(x)\rangle. \end{aligned}$$

□

Definition 3.3 The vector valued Wiener Amalgam space, denoted by $W(C, \ell^1, \mathbb{C}^m)$, is defined by

$$W(C, \ell^1, \mathbb{C}^m) = \{f \in C(\mathbb{R}, \mathbb{C}^m) : \sum_{n\in\mathbb{Z}} \max_{x\in[0,1]} ||f(x+n)|| < \infty\}.$$

If $\{\phi_1, \ldots\ldots, \phi_m\} \subset W(C, \ell^1, \mathbb{C}^m)$, then each function in $V(F)$ is continuous. Further it can be shown that $W(C, \ell^1, \mathbb{C}^m)$ is a Banach Space with the above norm.

Definition 3.4 A set $X = \{x_n : n \in \mathbb{Z}\}$ is called a *stable set* of sampling for $V(F)$, if there exist $r, R > 0$ such that

$$r \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} \parallel f(x_n) \parallel^2_{\mathbb{C}^m} \leq R \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)}, \forall f \in V(F). \tag{11}$$

Proposition 3.1 $X = \{x_n : n \in \mathbb{Z}\}$ is a stable set of sampling for $V(F)$ iff it is a stable set of sampling for $V(\phi_1), \ldots, V(\phi_m)$.

Proof Let X be a stable set of sampling for $V(\phi_i), i = 1, \ldots, m$. Then, there exist $r_i, R_i > 0$ such that

$$r_i \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} \parallel f_i(x_n) \parallel^2_{\mathbb{C}^m} \leq R_i \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)}, f_i \in V(\phi_i), \forall i = 1,\ldots,m. \tag{12}$$

Let $r = \underset{i=1,\ldots,m}{min}(r_i)$ and $R = \underset{i=1,\ldots,m}{max}(R_i)$. Let $f \in V(F)$. Then,

$$f = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{m} c_j(k)\phi_j(x-k)$$

for some $c_j \in l^2(\mathbb{Z})$. Here each $f_j = \sum_{k\in\mathbb{Z}} c_j(k)\phi_j(x-k) \in V(\phi_j), \quad \forall j = 1,\ldots,m.$ Then it follows from (12) that,

$$\sum_{i=1}^{m} r \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{i=1}^{m}\sum_{n\in\mathbb{Z}} r \parallel f_i(x_n) \parallel^2_{\mathbb{C}^m} \leq \sum_{i=1}^{m} R \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)}, \forall i = 1,\ldots,m.$$

which in turn implies that

$$\parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} r \parallel f(x_n) \parallel^2_{\mathbb{C}^m} \leq R \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)}.$$

Conversely, suppose X is a stable set of sampling for $V(F)$. Then, there exist $r, R > 0$ such that

$$r \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} \parallel f(x_n) \parallel^2_{\mathbb{C}^m} \leq R \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)}, \quad \forall f \in V(F).$$

Let $f_i \in V(\phi_i)$. Then,

$$f_i = \sum_{n\in\mathbb{Z}} c(n)\tau_n\phi_i \quad \text{where} \quad c \in \ell^2(\mathbb{Z}).$$

From the above equation, we get $f_i \in V(F)$. Therefore,

$$r \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} \parallel f_i(x_n) \parallel^2_{\mathbb{C}^m} \leq R \parallel f_i \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)},$$

which shows that X is a stable set of sampling for $V(\phi_i)$ for $i = 1,\ldots,m$. □

Definition 3.5 Let $f \in L^2(\mathbb{R},\mathbb{C}^m)$. Then, the vector valued Zak transform $Z : L^2(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R}\times\mathbb{T},\mathbb{C}^m)$ where $\mathbb{T} = [-1,1]$, is the function on $\mathbb{R}^2$ defined by

$$(Zf)(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y}\tau_n f(x). \tag{13}$$

As in the classical case, vector valued Zak transform satisfies the following properties.

- $Z\hat{f}(x,y) = e^{2\pi ixy}Zf(-y,x) \quad x,y \in \mathbb{R}.$
- $Zf(k,y) = e^{2\pi iky}Zf(0,y) \quad k \in \mathbb{Z}.$

Now, for each $\mathcal{F} \in L^2(\mathbb{T}, \mathbb{C}^m)$, we define

$$T\mathcal{F}(x) = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \langle \mathcal{F}, e^{2\pi in\cdot} e_l \rangle \phi_l(x+n),$$

where $(e_1, \ldots, e_m)$ is the standard orthonormal basis for $\mathbb{C}^m$.

Clearly $T\mathcal{F} \in V(F)$. The operator T is bounded above and below. In fact,

$$||T\mathcal{F}||^2 = ||\sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \langle \mathcal{F}, e^{2\pi in\cdot} e_l \rangle \phi_l(\cdot + n)||^2.$$

Since $\{\tau_n \phi_l; l = 1, \ldots, n\}$ is a Riesz Basis for $V(F)$, we have,

$$c \sum_n ||\langle \mathcal{F}, e^{2\pi in\cdot} e_l \rangle||^2 \leq ||T\mathcal{F}||^2 \leq \sum_n ||\langle \mathcal{F}, e^{2\pi in\cdot} e_l \rangle||^2,$$

which implies that

$$c||\mathcal{F}||^2 \leq ||T\mathcal{F}||^2 \leq C||\mathcal{F}||^2,$$

since $\{e^{2\pi in} e_l; l = 1, \ldots, m; n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T}, \mathbb{C}^m)$.

Moreover, for each $f \in V(F)$, there exists a unique $(c_n) \in \ell^2(\mathbb{Z})^m$ such that $f = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} c_j(n)\phi_l(\cdot + n)$. Again, by Riesz-Fischer theorem, there exist a unique $\mathcal{F} \in L^2(\mathbb{T}, \mathbb{C}^m)$, such that $\langle \mathcal{F}, e^{2\pi in} e_l \rangle = c_l(n)$. For this $\mathcal{F}$ we have $T\mathcal{F} = f$. Hence T is onto. Thus, T is a bounded invertible operator from $L^2(\mathbb{T}, \mathbb{C}^m)$ onto $V(F)$. Further,

$$\begin{aligned}
&T\mathcal{F}(x)\\
&= \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \langle \mathcal{F}, e^{2\pi in\cdot} e_l \rangle \phi_l(x+n)\\
&= \sum_{n\in\mathbb{Z}} \sum_{l=1}^{m} \int_T e^{-2\pi iny} \mathcal{F}_l(y) dy \phi_l(x+n)\\
&= \int_T \sum_{l=1}^{m} \mathcal{F}_l(y) \sum_{n\in\mathbb{Z}} e^{-2\pi iny} \phi_l(x+n) dy\\
&= \sum_{l=1}^{m} \int_T \mathcal{F}_l(y) Z\phi_l(x,y) dy\\
&= \sum_{l=1}^{m} \int_T \mathcal{F}_l(y) (Z\phi_{l,1}(x,y), \ldots, Z\phi_{l,m}(x,y))^T dy\\
&= \left(\sum_{l=1}^{m} \int_T \mathcal{F}_l(y) Z\phi_{l,1}(x,y) dy, \ldots, \sum_{l=1}^{m} \int_T \mathcal{F}_l(y) Z\phi_{l,m}(x,y) dy \right)\\
&= \left(\left\langle \mathcal{F}, (\overline{Z\phi_{1,1}(x,\cdot)}, \ldots, \overline{Z\phi_{m,1}(x,\cdot)}) \right\rangle, \ldots, \left\langle \mathcal{F}, (\overline{Z\phi_{1,m}(x,\cdot)}, \ldots, \overline{Z\phi_{m,m}(x,\cdot)}) \right\rangle \right)\\
&= \left(\left\langle \mathcal{F}, \overline{Z\phi_{\ldots,1}(x,\cdot)} \right\rangle, \ldots, \left\langle \mathcal{F}, \overline{Z\phi_{\cdot,m}(x,\cdot)} \right\rangle \right)
\end{aligned}$$

Now, let $f \in L^2(\mathbb{R}, \mathbb{C}^m)$ be written as $f = (f_1, \ldots, f_m)$. Consider

$$\sum_{k\in\mathbb{Z}}\sum_{j=1}^{m} c_j(k)\phi_j(x_i-k)=f(x_i),\quad c_j\in\ell^2(\mathbb{Z}), \tag{14}$$

which we will write as a matrix equation. Define $U:\ell^2(\mathbb{Z})^m\to\ell^2(\mathbb{Z})^m$ by

$$U=\begin{bmatrix} U_{1,1} & \dots & U_{m,1} \\ \vdots & \vdots & \vdots \\ U_{1,m} & \dots & U_{m,m}\end{bmatrix}, \tag{15}$$

where each $U_{i,j}$ is an infinite matrix given by

$$U_{i,j}(k,s)=\phi_{i,j}(x_k-s). \tag{16}$$

Let

$$c=\begin{bmatrix} c_1 \\ \vdots \\ c_m\end{bmatrix} \tag{17}$$

be an element of $l^2(\mathbb{Z})^m$, where each $c_i\in l^2(\mathbb{Z})$. Then (14) can be written as

$$Uc=(f(x_i)),\quad i\in\mathbb{Z}, \tag{18}$$

where

$$f=\begin{bmatrix} f_1 \\ \vdots \\ f_m\end{bmatrix} \tag{19}$$

belongs to $L^2(\mathbb{R},\mathbb{C}^m)$.

Now we shall prove the equivalent statements for a stable set of sampling for $V(F)$, as in [2, 26] for a shift invariant space in $L^2(\mathbb{R})$ with a single generator.

Theorem 3.6 *Let $F=\{\phi_1,\ldots\ldots,\phi_m\}\subset L^2(\mathbb{R},\mathbb{C}^m)$ and $F\subset W(C,\ell^1,\mathbb{C}^m)$ such that $\{\tau_k\phi_i, k\in\mathbb{Z}, i=1,\ldots,m\}$ is a Riesz basis for $V(F)$. Then the following statements are equivalent*:

(1) The set $X=\{x_j:j\in\mathbb{Z}\}$ is a stable set of sampling for $V(F)$.

(2) There exists $A,B>0$ such that

$$A\parallel c\parallel^2_{l^2(\mathbb{Z})^m}\ \le\ \parallel Uc\parallel^2_{l^2(\mathbb{Z})^m}\ \le B\parallel c\parallel^2_{l^2(\mathbb{Z})^m},\quad \forall c\in l^2(\mathbb{Z})^m.$$

(3) The set $\{K_{x_j}e_l:j\in\mathbb{Z}, l=1,\ldots,m\}$ is a super frame for $V(F)$, where $\{K_x : x\in\mathbb{R}\}$ is a reproducing kernel for $V(F)$.

(4) The set $\{Z\phi_{.,1}(x_j,.),\ldots,Z\phi_{.,m}(x_j,.):j\in\mathbb{Z}\}$ is a super frame for $L^2(\mathbb{T},\mathbb{C}^m)$.

Proof $1 \Rightarrow 2$. Let $X = \{x_j : j \in \mathbb{Z}\}$ be a stable set of sampling for $V(F)$. Then, there exist $r > 0, R > 0$ such that

$$r \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} \leq \sum_{n\in\mathbb{Z}} \parallel f(x_n) \parallel^2_{\mathbb{C}^m} \leq R \parallel f \parallel^2_{L^2(\mathbb{R},\mathbb{C}^m)} .$$

Let $f = \sum_{n\in\mathbb{I}} \sum_{j=1}^m c_j(n)\tau_n\phi_j$. Then,

$$\begin{aligned}
\parallel f \parallel^2 &= \left|\left\langle \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} c_i(n)\tau_n\phi_i, \sum_{k\in\mathbb{I}} \sum_{j=1}^{m} c_j(k)\tau_k\phi_j \right\rangle\right| \\
&= \left| \sum_{n,k\in\mathbb{Z}} \sum_{i,j=1}^{m} \langle c_i(n)\tau_n\phi_i, c_j(k)\tau_k\phi_j\rangle \right| \\
&\leq \sum_{n,k\in\mathbb{Z}} \sum_{i,j=1}^{m} |c_i(n)|||\tau_n\phi_i|||c_j(k)|||\tau_k\phi_j|| \\
&= \max_l \parallel \phi_j \parallel^2 \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} |c_i(n)| \sum_{k\in\mathbb{Z}} \sum_{j=1}^{m} |c_j(k)| \\
&\leq B \parallel c \parallel^2 .
\end{aligned}$$

Also,

$$\parallel f \parallel^2 \geq \min_l \parallel \phi_j \parallel^2 \sum_{n\in\mathbb{I}} \sum_{j=1}^{m} |c_j(n)|^2.$$

Further, we have from (18) that, $\parallel Uc \parallel^2_{l^2(\mathbb{Z})^m} = \sum_{n\in\mathbb{Z}} |f(x_n)|^2$.

Therefore,

$$A \parallel c \parallel^2_{l^2(\mathbb{Z})^m} \leq \parallel Uc \parallel^2_{l^2(\mathbb{Z})^m} \leq B \parallel c \parallel^2_{l^2(\mathbb{Z})^m}, \quad \forall c \in l^2(\mathbb{Z})^m,$$

where $A = r \cdot \min_l \parallel \phi_j \parallel^2, \quad B = R \cdot \max_l \parallel \phi_j \parallel^2$.

$2 \Rightarrow 3$. From (11),

$$\begin{aligned}
\sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle f, K_{x_j}e_l\rangle|^2 &= \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle f(x_j), e_l\rangle|^2 \\
&= \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|f_l(x_j)|^2 \\
&= \sum_{n\in\mathbb{I}} \| f(x_j) \|^2 \\
&= \| Uc \|^2 \\
&\leq B \| c \|^2 \\
&\leq B\frac{1}{A} \| f \|^2 .
\end{aligned}$$

Similarly,

$$A\frac{1}{B} \| f \|^2 \leq \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle f, K_{x_j}e_l\rangle|^2.$$

Notice here that, with $H_l = L^2(\mathbb{R}, \mathbb{C}^m), l = 1, \ldots, m$ and $\pi_l : \mathbb{Z} \to L^2(\mathbb{R}, \mathbb{C}^m)$ given by $\pi_l(j) = K_{x_j}e_l, l = 1, \ldots, m$, we have $\{\pi_l(j), l = 1, \ldots, m\}$ is a super frame for $V(F)$.

$3 \Longleftrightarrow 4$. We have,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \| f(x_n) \|^2_{\mathbb{C}^m} &= \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle f, K_{x_j}e_l\rangle|^2 \\
&= \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle \mathcal{F}, Z\phi_{\cdot,l}(x_j,\cdot)\rangle|^2,
\end{aligned}$$

where $T\mathcal{F} = f$ and T being a bounded invertible operator satisfies

$$\frac{1}{\| T^{-1} \|^2} \| \mathcal{F} \|^2 \leq \sum_{n\in\mathbb{I}}\sum_{j=1}^{m}|\langle \mathcal{F}, Z\phi_{\cdot,l}(x_j,\cdot)\rangle|^2 \leq \| T \|^2 \| \mathcal{F} \|^2,$$

showing that 3 and 4 are equivalent. □

4 Regular sampling in *V*(*F*)

Consider the case when $X = \mathbb{Z}$. Then the matrix U in (15) becomes

$$U_{i,j}(k, s) = \phi_{i,j}(k - s).$$

In this case, $U : \ell^2(\mathbb{Z})^m \longrightarrow \ell^2(\mathbb{Z})^m$ is a block Laurent operator by its structure. Then by (5), each of $U_{i,j} : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ are Laurent operators.

For $F = \{\phi_1, \ldots\ldots, \phi_m\}$, define Φ by

$$\Phi(z) = \begin{bmatrix} \Phi_{1,1}(z) & \ldots & \Phi_{m,1}(z) \\ \vdots & \vdots & \vdots \\ \Phi_{1,m}(z) & \ldots & \Phi_{m,m}(z) \end{bmatrix}, \tag{20}$$

where,

$$\Phi_{i,j}(z) = \sum_{n \in \mathbb{Z}} \phi_{i,j}(n) z^n \tag{21}$$

and $z \in S^1$, where $S^1 = \{z \in \mathbb{C}, |z| = 1\}$.

From [12], we have $\Phi_{i,j}$ is the defining function of the Laurent operator with matrix $U_{i,j}$. Therefore, from [12], we have Φ is the defining function of U. If $\phi_1, \ldots\ldots, \phi_m \in \mathbb{W}(C, \ell^1, \mathbb{C}^m)$, then each $\phi_{i,j} \in \mathbb{W}(C, \ell^1, \mathbb{C})$, and the infinite sum in (21) converges uniformly on S^1, from which we have $\Phi_{i,j}$ is continuous on S^1.

Theorem 4.1 *Suppose $\phi_1, \ldots\ldots, \phi_m \in \mathbb{W}(C, \ell^1)$. Then, the operator U described in 15 with $U_{i,j}(k, s) = \phi_{i,j}(k - s), k, s \in \mathbb{Z}$ is a Block Laurent operator on $l^2(\mathbb{Z})^m$ with symbol Φ. Moreover the operator U satisfies the inequalities*

$$||\Phi||_0^2 ||d||^2_{l^2(\mathbb{Z})^m} \leq ||Ud||^2_{l^2(\mathbb{Z})^m} \leq ||\Phi||^2_\infty ||d||^2_{l^2(\mathbb{Z})^m}, \tag{22}$$

$\forall d = (d_1, \ldots, d_m) \in l^2(\mathbb{Z})^m$, where $||\Phi||_\infty = \max_{S^1} ||\Phi(x)||_{\mathbb{HS}}$ and $||\Phi||_0 = \min_{S^1} ||\Phi(x)||_{\mathbb{HS}}$, where $||.||_{\mathbb{HS}}$ denotes the Hilbert Schmidt norm.

Proof

$$\begin{aligned} ||Ud||^2_{l^2(\mathbb{Z})^m} &= \sum_{k=1}^m || \sum_{j=1}^m U_{j,k} d_j ||^2 \\ &= \sum_{k=1}^m \sum_{n \in \mathbb{Z}} | \sum_{j=1}^m (U_{j,k} d_j)(n)|^2 \\ &= \sum_{k=1}^m \int_{\mathbb{S}^1} | \sum_{l \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{j=1}^m \phi_{j,k}(l - n) d_j(n) e^{2\pi i l x}|^2 dx \\ &= \sum_{k=1}^m \int_{\mathbb{S}^1} | \sum_{s \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{j=1}^m \phi_{j,k}(s) d_j(n) e^{2\pi i (s+n) x}|^2 dx \\ &= \sum_{k=1}^m \int_{\mathbb{S}^1} | \sum_{j=1}^m \sum_{s \in \mathbb{Z}} \phi_{j,k}(s) e^{2\pi i s x} \sum_{n \in \mathbb{Z}} d_j(n) e^{2\pi i n x}|^2 dx \\ &= \sum_{k=1}^m \int_{\mathbb{S}^1} | \sum_{j=1}^m \Phi_{j,k}(x) \sum_{n \in \mathbb{Z}} d_j(n) e^{2\pi i n x}|^2 dx. \end{aligned}$$

Let $f_j(x) = \sum_{n \in \mathbb{Z}} d_j(n) e^{2\pi i n x}$ and $f(x) = (f_1(x), \ldots, f_m(x))$. Then, by Parseval's

Identity, $||f_j||^2 = \sum_{n\in\mathbb{Z}} |d_j(n)|^2, j = 1, \ldots, m$ and hence $||f||^2 = ||d||^2$. Also denote $\Phi = [\Phi_1, \ldots, \Phi_m]^T$ where $\Phi_j, j = 1, \ldots, m$ are the rows of Φ. Then,

$$||Ud||^2_{l^2(\mathbb{Z})^m} = \sum_{k=1}^{m} \int_{\mathbb{S}^1} |\Phi_k(x)f(x)|^2 dx$$
$$= \int_{\mathbb{S}^1} ||\Phi(x)f(x)||^2_{\mathbb{C}^m} dx.$$

Then U satisfies 22. □

Corollary 4.1 *Let $F = \{\phi_1, \ldots\ldots, \phi_m\} \subset L^2(\mathbb{R}, \mathbb{C}^m)$, such that $\{\tau_k\phi_i : k \in \mathbb{Z}, i = 1, \ldots, m\}$ forms a Riesz basis for V(F) and $||\Phi||_0 > 0$. Then $\mathbb{Z}$ is a stable set of sampling for V(F) iff there exists $\gamma > 0$, such that*

$$|det(\Phi^*(z)\Phi(z))| \geq \gamma \quad a.e. z \in S^1. \tag{23}$$

Proof Let the sampling set be $X = \mathbb{Z}$. Then, U is a block Laurent operator with symbol Φ and hence, U^*U is a block Laurent operator with symbol $\Phi^*\Phi$. It follows from theorem 4.1 that U is bounded above and below, which is equivalent to U^*U invertible, which shows that $\mathbb{Z}$ is a stable set of sampling for $V(F)$. □

Example 4.2 Let f be defined by

$$f(x) = \begin{cases} 4 + 3x & -4/3 \leq x < 0 \\ 4 - 3x & 0 \leq x \leq 4/3 \\ 0 & otherwise \end{cases}.$$

Define $\phi_1 = (f, 0); \phi_2 = (0, f)$. Then $\phi_1, \phi_2 \in L^2(\mathbb{R}, \mathbb{R}^2)$. We have $B = \{\tau_n\phi_1, \tau_m\phi_2 : n, m \in \mathbb{Z}\}$ forms a Riesz basis for $V(F)$. In fact, we have $\{\tau_n\phi_i : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi_i), i = 1, 2$. Also $\langle\tau_n\phi_1, \tau_m\phi_2\rangle = 0 \quad for \quad n, m \in \mathbb{Z}$. Therefore, $V(F) = V(\phi_1) \oplus V(\phi_2)$ and thus B is a Riesz Basis for $V(F)$. The matrix of the corresponding Laurent operator is given by

$$U = \begin{bmatrix} U_{1,1} & U_{2,1} \\ U_{1,2} & U_{2,2} \end{bmatrix} : l^2(\mathbb{Z})^2 \longrightarrow l^2(\mathbb{Z})^2,$$

where $U_{2,1} = U_{1,2} = 0$. Further,

$$U_{1,1} = \begin{bmatrix} \ddots & & & & & \\ & 1 & 4 & 1 & 0 & 0 \\ & 0 & 1 & 4 & 1 & 0 \\ & 0 & 0 & 1 & 4 & 1 \\ & & & & & \ddots \end{bmatrix} = U_{2,2}$$

In this case we can see that U is invertible being diagonally dominant, hence U^*U and therefore $\mathbb{Z}$ is a stable set of sampling for f.

5 An illustration

We further illustrate regular sampling and reconstruction using Matlab by taking an example in $L^2(\mathbb{R}, \mathbb{R}^2)$. Define $\phi_1 = (\phi_{1,1}, \phi_{1,2})$ and $\phi_2 = (\phi_{2,1}, \phi_{2,2})$ where $\phi_{1,1} = e^{-x^2}, \phi_{1,2} = (4x^2 - 2)e^{-x^2}$. We take $\phi_{2,1} = \phi_{1,2}, \phi_{2,2} = \phi_{1,1}$. Let $f = (f_1, f_2) \in V(\phi_1, \phi_2)$ be a finite length signal in the interval $[-20, 20]$ defined by $f = \sum_{k=-15}^{k=15} \sum_{j=1}^{2} c_j(k)\tau_k\phi_j$, where c_1 and c_2 are defined by $c_1(-15) = -1, c_1(-13) = \frac{1}{2}, c_1(-11) = 1, c_1(-9) = 1, c_1(-8) = 1, c_1(-5) = \frac{-1}{2}, c_1(0) = 1, c_1(3) = 2, c_1(5) = \frac{-3}{2}, c_1(9) = 1, c_1(11) = -1, c_1(13) = -1, c_1(14) = \frac{-1}{2}$ and $c_2(-15) = -2, c_2(-13) = \frac{1}{2}, c_2(-11) = 1, c_2(-9) = 1, c_2(-8) = 1, c_2(-5) = \frac{-1}{2}, c_2(0) = 1, c_2(3) = \frac{3}{2}, c_2(5) = \frac{-1}{2}, c_2(9) = \frac{3}{2}, c_2(11) = -1, c_2(13) = -1, c_2(14) = 1\frac{1}{2}$. Also $c_1(i) = 0$ and $c_2(i) = 0$ for the remaining values of i.

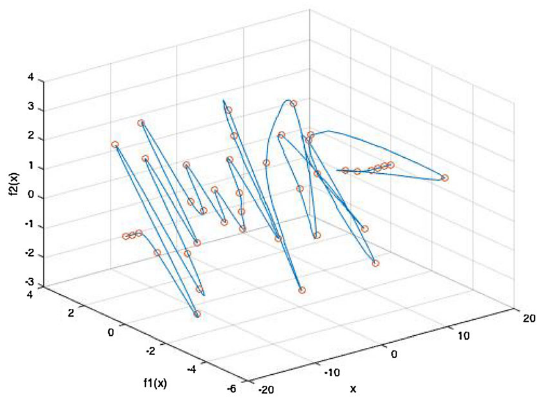
Here,

$$\Phi(z) = \begin{bmatrix} \Phi_{1,1}(z) & \Phi_{2,1}(z) \\ \Phi_{1,2}(z) & \Phi_{2,2}(z) \end{bmatrix},$$

where $\Phi_{1,1}(z) = \sum_{n \in \mathbb{Z}} e^{-n^2} z^n$ and $\Phi_{1,2}(z) = \sum_{n \in \mathbb{Z}} (4n^2 - 2)e^{-n^2} z^n$. Also $\Phi_{2,1} = \Phi_{1,2}, \Phi_{2,2} = \Phi_{1,1}$. In this case $||\Phi(z)||^2 = ||\Phi_{1,1}(z)||^2 + ||\Phi_{1,2}(z)||^2 + ||\Phi_{2,1}(z)||^2 + ||\Phi_{2,2}(z)||^2 = 2(||\Phi_{1,1}(z)||^2 + ||\Phi_{1,2}(z)||^2)$. It can be shown that $||\Phi_{1,1}(z)|| > \frac{3}{4} - \frac{2}{e} > 0$ for $z \in \mathbb{S}^1$. $\mathbb{S}^1$ being compact, we have $\min_{z \in \mathbb{S}^1} ||\Phi_{1,1}(z)|| > 0$, so that $||\Phi||_0 > 0$.

Figure 1 shows the original function together with its samples taken at integer points. Figure 2 shows the reconstructed function. The relative reconstruction error given by $\frac{||f_{org} - f_{recon}||_{L^2(\mathbb{R}, \mathbb{C}^m)}}{||f_{org}||_{L^2(\mathbb{R}, \mathbb{C}^m)}}$ is 1.2439×10^{-13}.

Fig. 1 f Original



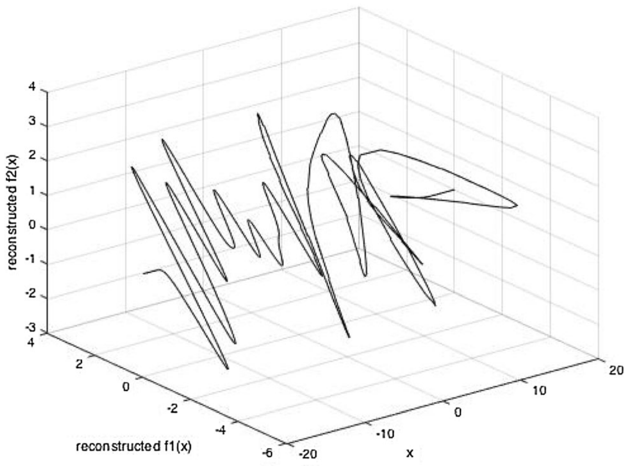


Fig. 2 f Reconstructed

6 A sampling formula

In this section, we obtain a sampling formula for functions in $V(F)$ with integer samples.

Theorem 6.1 *Let* $F = \{\phi_1, \ldots\ldots, \phi_m\} \subset \mathbb{W}(C, \ell^1, \mathbb{C}^m)$ *be such that* $\{\tau_n\phi_1, \ldots, \tau_n\phi_m : n \in \mathbb{Z}\}$ *forms a Riesz basis for* $V(F)$. *Suppose there exists a* $\gamma > 0$ *such that*

$$\{z \in S^1 : |det\Phi^{\dagger}(z)| < \gamma\} \tag{24}$$

has measure zero, where $\Phi^{\dagger}$ is the symbol of the block Laurent operator U^*U associated with the sample set $\mathbb{Z}$. Then, $\forall f \in V(\mathbb{F})$, we have

$$f(x) = \sum_{j,l=1}^{m} \sum_{k,n\in\mathbb{Z}} (U^{l*}Y)(n)\widehat{\psi}_{l,j}(k-n)\phi_j(x-k), \tag{25}$$

where $\Psi = [\psi_{i,j}]$ is the symbol associated with inverse block Laurent operator with sampling set $\mathbb{Z}$.

Proof For $f \in V(\mathbb{F})$, we have

$$f(x) = \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} c_j(k)\phi_j(x-k). \tag{26}$$

Writing U as $U = [U^1, \ldots, U^m]$ where each $U^i = [U_{i,1}, \ldots, U_{i,m}]^T$ and $c = (c_1 \ldots c_m)^T$, then for the sampling set $\mathbb{Z}$, we can write (26) as $Uc = Y$ where $Y = (f(x_s))_{x_s\in\mathbb{Z}}$. Then $U^*Uc = U^*Y$. Since for $\gamma > 0$ we have, $\{z \in S^1 : |det\Phi^{\dagger}(z)| < \gamma\}$ has measure zero, U^*U is invertible. As U^*U is a block Laurent operator with symbol $\Phi^{\dagger}$, $(U^*U)^{-1}$ will be a block Laurent Operator with symbol, say, $\Psi = [\psi_{j,i}]_{1\leq i,j\leq m}$ where $\Psi = (\Phi^{\dagger})^{-1}$. Let

$$(U^*U)^{-1} = L = [L_{j,i}]_{1\leq} i,j \leq m. \tag{27}$$

Then,

$$c_p = \sum_{i=1}^{m}\sum_{l=1}^{m} L_{l,p}U^*_{l,i}f_i = \sum_{l=1}^{m} L_{l,p}U^{l*}Y.$$

Hence,

$$\begin{aligned} f(x) &= \sum_{j=1}^{m}\sum_{k\in\mathbb{Z}}\left(\sum_{l=1}^{m} L_{l,j}U^{l*}Y\right)_k \phi_j(x-k) \\ &= \sum_{j=1}^{m}\sum_{k\in\mathbb{Z}}\left(\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}} (L_{l,j})_{k,n}(U^{l*}Y)(n)\right)_k \phi_j(x-k) \\ &= \sum_{j=1}^{m}\sum_{k\in\mathbb{Z}}\left(\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}} \widehat{\psi}_{l,j}(k-n)(U^{l*}Y)(n)\right)_k \phi_j(x-k) \\ &= \sum_{j,l=1}^{m}\sum_{k,n\in\mathbb{Z}} (U^{l*}Y)(n)\widehat{\psi}_{l,j}(k-n)\phi_j(x-k). \end{aligned}$$

□

7 Perturbation of a stable set of sampling

In this section we prove that if a set Y is "sufficiently close" to a stable sampling set X, then Y will also turn out to be a stable set of sampling for $V(F)$. For a similar theorem in the classical case we refer to [15, 24].

Definition 7.1 For $\delta > 0$, a set $Y = \{y_j : j \in \mathbb{Z}\}$ is said to be in a δ - neighbourhood of $X = \{x_j : j \in \mathbb{Z}\}$, if $|x_j - y_j| < \delta, \forall j \in \mathbb{Z}$.

Let

$$U^X = \begin{bmatrix} U^X_{1,1} & \cdots & U^X_{m,1} \\ \vdots & \vdots & \vdots \\ U^X_{1,m} & \cdots & U^X_{m,m} \end{bmatrix} \tag{28}$$

and $||U^X|| = sup_{1\leq i,j\leq m}||U^X_{i,j}||_{op}$. For each $i,j \in 1,\ldots,m$, let $U^X_{i,j}$ be a bounded operator on $\ell^2(\mathbb{Z})$ where $U^X_{i,j}$ is defined by the infinite matrix

$$U^X_{i,j}(k,s) = \phi_{i,j}(x_k - s).$$

Theorem 7.2 *Let ϕ_j for each $j = 1, \ldots, m$ be a continuous $\mathbb{C}^m$ valued function having compact support in $\mathbb{R}$. Then $\forall \epsilon > 0$, there exists $\delta > 0$ such that whenever Y is in a δ-neighbourhood of X, $||U^X - U^Y|| < \epsilon$.*

Proof Let $\phi_j = (\phi_{j,1}, \ldots, \phi_{j,m})$. Let $i \in \{i, \ldots, m\}$. Since ϕ_j has compact support in $\mathbb{R}, \phi_{j,i}$ also has compact support in $\mathbb{R}$. Also $\phi_{j,i} : j = 1, \ldots, m$ is continuous. Therefore, there exists an integer $N \in \mathbb{N}$ such that

$$\sum_{k \in \mathbb{Z}} |\phi_{j,i}(x_l - k) - \phi_{j,i}(y_l - k)| = \sum_{||k|| \leq N} |\phi_{j,i}(x_l - k) - \phi_{j,i}(y_l - k)|. \tag{29}$$

Here ||*k*|| denotes the Euclidean norm of k. Being continuous over a compact support, $\phi_{j,i}$ is uniformly continuous in $\mathbb{R}$. Therefore given $\epsilon > 0$, there exist a $\delta > 0$, such that

$$|\phi_{j,i}(x) - \phi_{j,i}(y)| < \frac{\epsilon}{\#(k : ||k|| < N)}, \quad \text{whenever} \quad ||x - y|| < \delta.$$

Suppose Y is in this δ- neighbourhood. Then $||x_l - y_l|| < \delta$ for every $l \in \mathbb{Z}$. Hence,

$$\sum_{k \in \mathbb{Z}} |\phi_{j,i}(x_l - k) - \phi_{j,i}(y_l - k)| < \epsilon, \quad \forall l \in \mathbb{Z}.$$

This implies that,

$$sup_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\phi_{j,i}(x_l - k) - \phi_{j,i}(y_l - k)| < \epsilon. \tag{30}$$

Similarly, it can be shown that

$$sup_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\phi_{j,i}(x_k - l) - \phi_{j,i}(y_k - l)| < \epsilon. \tag{31}$$

Therefore , by Schur's test, for any bounded operator A on $\ell^2(\mathbb{Z})$ described by a matrix $[a_{k,l}]$ we have

$$||A|| \leq \sqrt{\alpha\beta},$$

where $\alpha = sup_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |a_{k,l}|$ and $\beta = sup_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |a_{k,l}|$. Then, using (30) and (31), we get $||U^X_{i,j} - U^Y_{i,j}|| < \epsilon$. Therefore, $||U^X - U^Y|| < \epsilon$ being supremum of the individual norms for $i, j \in \{1, \ldots, m\}$. □

Definition 7.3 A sampling set $X = \{x_i : i \in \mathbb{Z}\}$ is said to be separated by δ, if $sep(X) = inf_{k \neq l}|x_k - x_l| \geq \delta$ for some $\delta > 0$.

Theorem 7.4 *Let ϕ_j for each $j = 1, \ldots, m$ be continuous $\mathbb{C}^m$ valued functions on $\mathbb{R}$ satisfying the decay condition $||\phi_j(x)|| \leq \frac{c}{1+|x|^\alpha}, \forall x \in \mathbb{R}, j = 1, \ldots, m$ where $c > 0$ is a constant and $\alpha > 1$. Further assume that $\{\tau_k \phi_j : j = 1, \ldots, m\}$ is a Riesz basis for $V(F)$. Suppose $X = \{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(F)$ and it is* *separated by* δ_0 *for some* $\delta_0 > 0$. *Then there exists* $0 < \delta < \frac{\delta_0}{4}$ *such that whenever Y is in a* δ *neighbourhood of X, Y will be a stable set of sampling for* $V(F)$.

In order to prove the theorem, we prove the following.

Lemma 7.5 *Let* ϕ_j, *for* $j = 1, \ldots, m$ *be continuous* $\mathbb{C}^m$ *valued functions on* $\mathbb{R}$ *satisfying the decay condition* $||\phi_j(x)|| \leq \frac{c}{1+|x|^\alpha}, \forall x \in \mathbb{R}, j = 1, \ldots, m$ *where* $c > 0$ *is a constant and* $\alpha > 1$. *Further, assume that* $\{\tau_k \phi_j : j = 1, \ldots, m\}$ *is a Riesz Basis for* $V(F)$. *Then, for any sampling set* $X = \{x_i : i \in \mathbb{Z}\}$, *the family of operators* $\{U^X : sep(X) \geq \delta\}$ *is uniformly bounded.*

Proof Consider $U^X_{i,j} : i, j \in \{1, \ldots, m\}$. We will show that the row and column sum of $U^X_{i,j}$ is bounded for every X with $sep(X) \geq \delta$.

We have,

$$\sum_{k \in \mathbb{Z}} |\phi_{i,j}(x_n + k)| \leq \sum_{k \in \mathbb{Z}} \frac{c}{1 + |x_n + k|^\alpha}, \forall n \in \mathbb{Z}. \tag{32}$$

The RHS converges uniformly . Therefore, the LHS also converges uniformly to a continuous function.

Hence we have,

$$\sum_{k \in \mathbb{Z}} |\phi_{i,j}(x_n + k)| \leq \alpha, \forall n \in \mathbb{Z}, \alpha < \infty.$$

Thus,

$$\sum_{k \in \mathbb{Z}} |\phi_{i,j}(x_n - k)| \leq \alpha, \forall n \in \mathbb{Z}, \alpha < \infty. \tag{33}$$

Also, since X is separated by δ for each $k \in \mathbb{Z}$, there exists an $n_0 \in \mathbb{Z}$ such that $|x_n - k| > \delta(n - n_0), \forall n \in \mathbb{Z}$. Then for each $k \in \mathbb{Z}$ we have,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} |\phi_{i,j}(x_n - k)| &\leq \sum_{n \in \mathbb{Z}} \frac{c}{1 + |x_n + k|^\alpha} \\
&\leq \sum_{n \geq n_0} \frac{c}{1 + \delta^\alpha (n - n_0)^\alpha} + \sum_{n < n_0} \frac{c}{1 + \delta^\alpha (n_0 - n)^\alpha} \\
&= \sum_{n=0}^{\infty} \frac{c}{1 + \delta^\alpha (n)^\alpha} + \sum_{n=0}^{\infty} \frac{c}{1 + \delta^\alpha (n)^\alpha} \\
&= 2c \sum_{n=0}^{\infty} \frac{c}{1 + (\delta n)^\alpha} < \infty.
\end{aligned}$$

Thus,

$$\sum_{n\in\mathbb{Z}} |\phi_{i,j}(x_n - k)| \le \beta_\delta, \forall k \in \mathbb{Z} \tag{34}$$

Hence, by (33) and (34), by Schur's lemma we have $||U^X|| \le \sqrt{\alpha\beta_\delta}$. Thus, the family of operators $\{U^X : sep(X) \ge \delta\}$ is uniformly bounded. □

Proof of theorem 7.4 Let X be a stable set of sampling with $sep(X) = \delta_0$. Let Y be in a $\frac{\delta_0}{4}$ neighbourhood of X.

We have for $k, l \in \mathbb{Z}$,

$$\begin{aligned} |x_k - x_l| &= |x_k - y_k + y_k - y_l + y_l - x_l|, k, l \in \mathbb{Z} \\ &\le |x_k - y_k| + |y_k - y_l| + |y_l - x_l| \\ &\le \frac{\delta_0}{2} + |y_k - y_l|. \end{aligned}$$

Thus,

$$\begin{aligned} |y_k - y_l| &\ge |x_k - x_l| - \frac{\delta_0}{2} \\ &\ge \delta_0 - \frac{\delta_0}{2} \\ &= \frac{\delta_0}{2}. \end{aligned}$$

In other words, $sep(Y) \ge \frac{\delta_0}{2}$. From lemma 33 , we have $||U^Y|| \le M$ for all operators U^Y in $\{U^Y : sep(Y) \ge \frac{\delta_0}{2}\}$ for some $M > 0$. In particular, $||U^X|| \le M$, since X is a stable set of sampling with $sep(X) = \delta_0$. By theorem 7.2, for $\epsilon = \frac{1}{2M||(U^{X*}U^X)^{-1}||}$, there exist $0 < \delta \le \frac{\delta_0}{4}$ such that $||U^X - U^Y|| < \epsilon$ whenever Y is in a δ-neighbourhood of X.

$$\begin{aligned} ||U^{X*}U^X - U^{Y*}U^Y|| &= ||U^{X*}U^X - U^{X*}U^Y + U^{X*}U^Y - U^{Y*}U^Y|| \\ &\le ||U^{X*}U^X - U^{X*}U^Y|| + ||U^{X*}U^Y - U^{Y*}U^Y|| \\ &= ||U^{X*}||||U^X - U^Y|| + ||U^Y||||U^{X*} - U^{Y*}|| \\ &< \frac{1}{||(U^{X*}U^X)^{-1}||}. \end{aligned}$$

Thus $U^{Y*}U^Y$ is invertible, whenever Y is in a δ neighbourhood of X, proving our assertion. □

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