GENERALIZED INVERSES AND APPROXIMATION NUMBERS

K.P. DEEPESH, S.H. KULKARNI, AND M.T. NAIR

Abstract. We derive estimates for approximation numbers of bounded linear operators between normed linear spaces. As special cases of our general results, approximation numbers of some weighted shift operators on ℓ^p and those of isometries and projections of norm 1 are found. In the case of finite rank operators, we obtain estimates for the smallest nonzero approximation number in terms of their generalized inverses. Also proved are some results regarding the relation between approximation numbers and the closedness of the range of an operator. It may be recalled that the closedness of the range is a necessary condition for the boundedness of a generalized inverse. Examples are given to illustrate the results, and also to show that certain inequalities need not hold.

1. Introduction

Let X and Y be normed linear spaces and $BL(X,Y)$ be the class of all bounded linear operators from X to Y. We use the notations $BL(X)$ for $BL(X,X)$ and X' for $BL(X,\mathbb{C})$. We shall denote the set of all finite rank operators $F \in BL(X,Y)$ with $\text{rank}(F) < k$ by $\mathcal{F}_k(X,Y)$ and use the notation $\mathcal{F}_k(X)$ for $\mathcal{F}_k(X,X)$. Also we denote by $\ell^p(n)$ the space $\mathbb{C}^n$ with the norm $\|\cdot\|_p$, $1 \le p \le \infty$. We shall also use the notation $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$ for $i, j \in \mathbb{N}$, and for $T \in BL(X,Y)$, the range of T is denoted by $R(T)$.

The concept of approximation numbers of operators in $BL(X,Y)$ is a generalization of the concept of singular values of compact operators between Hilbert spaces. For $T \in BL(X,Y)$ and $k \in \mathbb{N}$, the k^{th} **approximation number** $s_k(T)$ of T is defined as

$$s_k(T) := \inf\{\|T - F\| : F \in \mathcal{F}_k(X,Y)\}.$$

It is clear that $\|T\| = s_1(T) \ge s_2(T) \ge \ldots \ge 0$ and if T is of finite rank, then $s_k(T) = 0$ for all $k > \text{rank}(T)$.

Some studies on approximation numbers and their properties can be found in [14]. Approximation numbers play an important role in the geometry of Banach spaces as they are used in defining certain subclasses(*ideals*) of operator spaces ([15]). The convergence properties of approximation numbers are found useful in estimating the error while solving operator equations ([17]).

1991 *Mathematics Subject Classification.* Primary 47B06; Secondary 47A58.

Key words and phrases. Approximation numbers, Projection operators, Closed range operators, Generalized inverses.

K. P. Deepesh acknowledges the financial support received from the Council of Scientific and Industrial Research(CSIR), Government of India(File No: 09/084(0391)/2006-EMR-I).

Computation of approximation numbers is a very difficult task, even in the case of operators between finite dimensional spaces. There have been very few attempts in literature to estimate the approximation numbers of bounded linear operators between normed linear spaces. For example, [7] and [14] contain methods of computing approximation numbers of diagonal operators in $BL(\ell^q, \ell^p)$, $1 \leq p \leq q \leq \infty$, with non-increasing positive diagonal entries, diagonal operators between some finite dimensional spaces and embedding maps in $BL(\ell^p, \ell^q)$, $1 \leq p \leq q \leq \infty$. In [9] and [10], some estimates were given for approximation numbers of certain classes of integral operators.

The purpose of this article is to give some estimates for approximation numbers of bounded linear operators between normed linear spaces. Since, for a given $T \in BL(X,Y)$ and $k \in \mathbb{N}$, $s_k(T) \leq \|T - F\|$ for each operator $F \in \mathcal{F}_k(X,Y)$, finding lower estimates of approximation numbers is of importance. We give some results in this regard in Section 2. Approximation numbers of isometries, projections of norm 1, and that of some weighted shift operators in $BL(\ell^p)$, $1 \leq p \leq \infty$, are specified in this section. For finite rank operators in $BL(X,Y)$, we give an estimate for the least nonzero approximation number in terms of generalized inverses of the operator, and as a special case, we show that it coincides with the reciprocal of the norm of the Moore-Penrose inverse of the operator when X and Y are Hilbert spaces. This special case is a known result.

Let X, Y be Hilbert spaces, $T \in BL(X,Y)$, and let $T^* \in BL(Y,X)$ be the adjoint operator of T. In [8], it was shown that the closedness of $R(T)$, the range of T, can be characterized using the spectrum of T^*T. A question of interest is whether it is possible to study the closedness of $R(T)$ using $\{s_k(T)\}$, when X and Y are general normed linear spaces. It is relevant to note here that if T has a bounded generalized inverse, then $R(T)$ is closed and when X, Y are Hilbert spaces, then $T^\dagger$ is bounded if and only if $R(T)$ is closed [1]. In Section 3, we prove some results regarding the relation between $\{s_k(T)\}$ and closedness of $R(T)$ for $T \in BL(X,Y)$. We also give counter examples to show the inefficiency of approximation numbers in characterizing the closedness of $R(T)$.

2. Some estimates for approximation numbers

The following elementary proposition is useful to identify approximation numbers of some operators.

Proposition 2.1. *Let X, X_1, Y, Y_1 be normed linear spaces and $T \in BL(X,Y)$. Let $U \in BL(Y, Y_1)$ and $V \in BL(X_1, X)$ be surjective isometries. Then*

$$s_k(UTV) = s_k(T) \text{ for all } k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$. Then

$$s_k(UTV) \leq \|U\| s_k(T) \|V\| = s_k(T).$$

Also

$$s_k(T) = s_k(U^{-1}UTVV^{-1}) \leq \|U^{-1}\| s_k(UTV) \|V^{-1}\| = s_k(UTV).$$

Hence $s_k(UTV) = s_k(T)$ for all $k \in \mathbb{N}$. □

Example 2.2. Let $1 \leq p \leq \infty$ and $D \in BL(\ell^p(n))$ be the diagonal operator defined by

$$D(x_1, x_2, \dots, x_n) = (\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n), \quad (x_1, x_2, \dots, x_n) \in \ell^p(n),$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers satisfying $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$. Then it is known that $s_k(D) = \alpha_k$ for all $k \in \{1, 2, \dots, n\}$ (See [15]).

Now, suppose that $(a_1, a_2, \dots, a_n)$ is a rearrangement of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $A \in BL(\ell^p(n))$ is defined by

$$A(x_1, x_2, \dots, x_n) = (a_1 x_1, a_2 x_2, \dots, a_n x_n), \quad (x_1, x_2, \dots, x_n) \in \ell^p(n).$$

Since $A = UDV$ for appropriate isometries U and V, by Proposition 2.1 we have $s_k(A) = \alpha_k$ for $k = 1, \dots, n$. Also, if $B \in BL(\ell^p(n))$ is defined by

$$B(x_1, x_2, \dots, x_n) = (\alpha_1 x_2, \alpha_2 x_3, \dots, \alpha_{n-1} x_n, \alpha_n x_1), \; (x_1, x_2, \dots, x_n) \in \ell^p(n),$$

then $s_k(B) = \alpha_k$ for $k = 1, \dots, n$. To see this, first we observe that $B = UC$, where C is the diagonal operator defined by

$$C(x_1, x_2, \dots, x_n) = (\alpha_n x_1, \alpha_1 x_2, \dots, \alpha_{n-1} x_n), \quad (x_1, x_2, \dots, x_n) \in \ell^p(n),$$

and U is the surjective isometry on $\mathbb{C}^n$ defined by

$$U(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n, x_1), \quad (x_1, x_2, \dots, x_n) \in \ell^p(n).$$

Now, by Proposition 2.1, taking $V = I$, $s_k(B) = s_k(UC) = s_k(C)$ and since $(\alpha_n, \alpha_1, \dots, \alpha_{n-1})$ is a rearrangement of $(\alpha_1, \alpha_2, \dots, \alpha_n)$, $s_k(C) = s_k(D) = \alpha_k$. Since B is a compact operator, we also have $s_k(B') = \alpha_k$ (cf. [7]), where $B' \in BL(\ell^p)$ is the operator defined by

$$B'(x_1, x_2, \dots, x_n) = (\alpha_n x_n, \alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}), \quad (x_1, x_2, \dots, x_n) \in \ell^p.$$

In literature, very little is known about approximation numbers of general bounded linear operators, though approximation numbers of compact operators on Hilbert spaces (known as singular values), approximation numbers of diagonal operators in $BL(\ell^p, \ell^q)$ for $1 \leq q \leq p \leq \infty$ are known ([7, 14]). Also, some estimates are given for approximation numbers of certain classes of integral operators in [9] and [10].

Applicability of Proposition 2.1 is very limited since there may not be many isometries with the help of which one can transform a given operator in to a simpler form whose approximation numbers are known. For example, even for an operator $T \in BL(\ell^p(n))$ with $p \neq 2$, the class of operators T for which there exist a diagonal operator B and surjective isometries U, V such that $T = UBV$ is not very large. In fact, for $n = 2$ and $p \neq 2$, the operators for which this is possible are operators having matrix representations $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ or $\begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$, with respect to the standard basis of $\mathbb{C}^n$ ([2]). Note that both these cases are already covered in Example 2.2.

Let $T \in BL(X, Y)$ and for each $n \in \mathbb{N}$, let $P_n \in BL(X)$ and $Q_n \in BL(Y)$ be projections with $rank(P_n) = rank(Q_n) = n$ and $\|P_n\|\|Q_n\| = 1$. Let $T_n := Q_n T P_n$, $n \in \mathbb{N}$. We have proved in [5] (See Theorem 3.3 in [5]) that if X is separable, Y is the dual space of a separable normed linear space and if $T_n x \to Tx$ as $n \to \infty$ for each $x \in X$ in the weak* sense of convergence, then $\lim\limits_{n\to\infty} s_k(T_n) = s_k(T)$. In applications, the projections P_n and Q_n may be of finite rank, and one may know approximation numbers of the operators

$$\tilde{T}_n := T_n |_{R(P_n)} : R(P_n) \to R(Q_n).$$

So, a natural question is whether $s_k(T_n) = s_k(\tilde{T}_n)$ for $n, k \in \mathbb{N}$. We answer this question affirmatively, as a consequence of the following two propositions.

Proposition 2.3. *Let $T \in BL(X,Y)$, X_0 be a nonzero subspace of X, Y_0 be a nonzero subspace of Y such that $R(T) \subseteq Y_0$, $T_0 := T |_{X_0} : X_0 \to Y$ and $T_1 := T : X \to Y_0$. Then*

$$s_k(T_0) \leq s_k(T) \leq s_k(T_1).$$

Proof. Let $I_0 : X_0 \to X$ and $I_1 : Y_0 \to Y$ be the inclusion operators. Then $T_0 = TI_0$ and $T = I_1 T_1$ and hence

$$s_k(T_0) = s_k(TI_0) \leq s_k(T)\|I_0\| = s_k(T),$$

$$s_k(T) = s_k(I_1 T_1) \leq \|I_1\| s_k(T_1) = s_k(T_1).$$

□

In the next proposition, we use the notation $\widehat{T}$ to represent an operator $T \in BL(X,Y)$, considered as from X to $R(T)$, that is, $\widehat{T} := T : X \to R(T)$ defined by $\widehat{T}x = Tx$, $x \in X$.

Proposition 2.4. *Let $T \in BL(X,Y)$, and let $P \in BL(X)$ and $Q \in BL(Y)$ be nonzero projections. Then we have the following.*

(a) $\frac{1}{\|P\|} s_k(TP) \leq s_k(TP |_{R(P)}) \leq s_k(TP)$.
(b) $s_k(QT) \leq s_k(\widehat{QT}) \leq \|Q\| s_k(QT)$.
(c) $\frac{1}{\|P\|} s_k(QTP) \leq s_k(\widehat{QTP} |_{R(P)}) \leq \|Q\| s_k(QTP)$.

In particular, if $\|P\| = 1 = \|Q\|$, then

$$s_k(QTP) = s_k(\widehat{QTP} |_{R(P)}).$$

Proof. (a). We have

$$s_k(TP) = s_k(TP |_{R(P)} P) \leq \|P\| s_k(TP |_{R(P)})$$

and from Proposition 2.3, $s_k(TP |_{R(P)}) \leq s_k(TP)$.

(b). The inequality $s_k(QT) \leq s_k(\widehat{QT})$ follows from Proposition 2.3. For $F \in \mathcal{F}_k(X,Y)$, we have

$$\|\widehat{QT} - \widehat{QF}\| = \|QT - QF\| \leq \|Q\| \|QT - F\|.$$

Hence $s_k(\widehat{QT}) \leq \|Q\| \|QT - F\|$. Taking infimum over $F \in \mathcal{F}_k(X,Y)$, we get $s_k(\widehat{QT}) \leq \|Q\| s_k(QT)$.

(c). Taking $TP |_{R(P)}$ in place of T in (b) and using (a), we get

$$\begin{aligned} s_k(QTP |_{R(P)}) &\leq s_k(\widehat{QTP} |_{R(P)}) \\ &\leq \|Q\| s_k(QTP |_{R(P)}) \\ &\leq \|Q\| s_k(QTP). \end{aligned}$$

Also by taking QT in place of T in (a), $\frac{1}{\|P\|}s_k(QTP) \leq s_k(QTP\,|_{R(P)})$. Hence

$$\frac{1}{\|P\|}s_k(QTP) \leq s_k(\widehat{QTP}\,|_{R(P)}) \leq \|Q\|s_k(QTP).$$

The particular case is obvious from (c). □

The particular case in Proposition 2.4 together with Theorem 3.3 in [5] lead to the following.

Corollary 2.5. *Let X be separable, Y be the dual space of a separable space and $T \in BL(X,Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be sequences of projection operators in $BL(X)$ and $BL(Y)$, respectively, such that $\|P_n\| = 1 = \|Q_n\|$, $n \in \mathbb{N}$. Let*

$$T_n := Q_nTP_n \quad \text{and} \quad \tilde{T}_n := T_n\,|_{R(P_n)}: R(P_n) \to R(Q_n), \quad n \in \mathbb{N}.$$

If $T_nx \to Tx$ as $n \to \infty$ for each $x \in X$ in the weak sense of convergence, then for each $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(\tilde{T}_n) = s_k(T).$$

The above corollary helps us in identifying the approximation numbers of certain weighted shift operators on ℓ^p, $1 \leq p \leq \infty$, as illustrated in the following proposition.

Proposition 2.6. *Let $1 \leq p \leq \infty$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_i \geq \alpha_{i+1} \geq 0$ for every $i \in \mathbb{N}$. Let $A \in BL(\ell^p)$ be the operator defined by*

$$Ax = (\alpha_1x_2, \alpha_2x_3, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Then $s_k(A) = \alpha_k$ for every $k \in \mathbb{N}$.

Proof. For $n \in \mathbb{N}$, let $P_n \in BL(\ell^p)$ be the projection operator defined by

$$P_nx = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

and let $A_n := P_nAP_n$ and $\tilde{A}_n := A_n\,|_{R(P_n)}: R(P_n) \to R(P_n)$. Since $A_n \to A$ as $n \to \infty$ in the weak* operator topology (considering ℓ^p as a dual space), by Corollary 2.5, we have $s_k(\tilde{A}_n) \to s_k(A)$ as $n \to \infty$. Now the operator $\tilde{A}_n : \ell^p(n) \to \ell^p(n)$, defined by

$$\tilde{A}_nx = (\alpha_1x_2, \alpha_2x_3, \ldots, \alpha_{n-1}x_n, 0), \quad x = (x_1, x_2, \ldots, x_n) \in \ell^p(n)$$

can be obtained from the diagonal operator $B \in BL(\ell^p(n))$, defined by

$$Bx = (\alpha_1x_1, \alpha_2x_2, \ldots, \alpha_{n-1}x_{n-1}, 0), \quad x = (x_1, x_2, \ldots, x_n) \in \ell^p(n),$$

by composing with suitable isometries. Since approximation numbers of an operator do not change if the operator is composed with surjective isometries, we have

$$s_k(\tilde{A}_n) = s_k(B) = \alpha_k \text{ for all } k = 1, 2, \ldots, n-1.$$

Hence

$$s_k(A) = \lim_{n\to\infty} s_k(\tilde{A}_n) = \alpha_k \quad \forall k \in \mathbb{N}$$

□

Remark 2.7. Let $1 \le p \le \infty$ and $\{\alpha_n\}$ be as in Proposition 2.6. Let $T, S \in BL(\ell^p)$ be defined by

$$Tx = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

$$Sx = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p$$

respectively. Using similar arguments as in Proposition 2.6, it can be shown that

$$s_k(T) = \alpha_k = s_k(S) \quad \forall k \in \mathbb{N}.$$

Remark 2.8. It can be seen that $s_k(A)$ in Proposition 2.6 can also be found from the known values of $s_k(T)$(See [14]) by using the inequalities

$$s_k(T) = s_k(TV_-V_+) \le s_k(TV_-) \le s_k(T),$$

where V_- and V_+ are the left and right shift operators on ℓ^p, and $A = TV_-$. But we give Proposition 2.6 as an application of Corollary 2.5.

The following theorem helps in finding lower bounds for approximation numbers in certain cases.

Theorem 2.9. *Let $T \in BL(X,Y)$ and M be a subspace of X. Suppose there exists $\alpha > 0$ such that*

$$\|Tx\| \ge \alpha \|x\| \quad \forall x \in M.$$

Then

$$s_k(T) \ge \alpha \quad \forall\, k \le \dim(M).$$

In particular, if M is infinite dimensional, then

$$s_k(T) \ge \alpha \quad \forall\, k \in \mathbb{N}.$$

Proof. Suppose $k \in \mathbb{N}$ with $k \le \dim(M)$, and $F \in \mathcal{F}_k(X,Y)$. Then, it follows that $N(F\,|_M) \ne \{0\}$, so that there exists $x \in M$ with $\|x\| = 1$ and $F(x) = 0$. Therefore,

$$\alpha \le \|Tx\| = \|Tx - Fx\| \le \|T - F\|$$

This is true for all $F \in \mathcal{F}_k(X,Y)$. Hence $\alpha \le s_k(T)$. □

Remark 2.10. Suppose there exists an $\alpha > 0$ and $k \in \mathbb{N}$ such that $s_k(T) \ge \alpha$. One may ask whether there exists a subspace M of X such that $\dim(M) \ge k$ and $\|Tx\| \ge \alpha\|x\|$ for all $x \in M$. The answer is negative. To see this, consider the inclusion operator $I : \ell^2 \to \ell^\infty$. Then $s_k(I) = 1$ for all $k \in \mathbb{N}$ ([7]). Now assume that there exists a subspace M of ℓ^2 such that $\dim(M) \ge 2$ and $\|Ix\|_\infty \ge 1\,\|x\|_2$ for all $x \in M$. Since $\|x\|_\infty \le \|x\|_2$ for all $x \in \ell^2$, it follows that $\|x\|_\infty = \|x\|_2$ for all $x \in M$. This implies that for each $x \in M$, there exists $\beta \in \mathbb{C}$ and $j \in \mathbb{N}$ such that $x = \beta e_j$, where $e_j = (\delta_{jn})$, $j \in \mathbb{N}$. This j is independent of x. To see this, suppose $x, y \in M$ given by $x = \beta_1 e_{j_1}$ and $y = \beta_2 e_{j_2}$ for some nonzero $\beta_1, \beta_2 \in \mathbb{C}$ and $j_1, j_2 \in \mathbb{N}$. Then, since $x + y \in M$, we get $e_{j_1} = e_{j_2}$ so that $\dim(M) = 1$, which is a contradiction to our assumption that $\dim(M) \ge 2$. Thus it is impossible to have the relation $\|x\|_\infty \ge \|x\|_2$ for all $x \in M$, if $\dim(M) \ge 2$.

Corollary 2.11. *Let $T \in BL(X,Y)$ be bounded below. Then $T^{-1} : R(T) \to X$ is continuous and*

$$s_k(T) \ge \frac{1}{\|T^{-1}\|} \quad \forall\, k \le \operatorname{rank}(T).$$

In particular, if T is an isometry, then $s_k(T) = 1$ for all $k \le \operatorname{rank}(T)$.

Proof. Since T is bounded below, T is injective, $\operatorname{rank}(T) = \dim(X)$ and $T^{-1} : R(T) \to X$ is continuous. In particular,

$$\|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\| \, \|Tx\| \quad \forall\, x \in X.$$

Hence, by Theorem 2.9, we have $s_k(T) \geq \frac{1}{\|T^{-1}\|}$ for all $k \leq \dim(X)$. The particular case is obvious. □

Remark 2.12. By Corollary 2.11, we can infer that if X is infinite dimensional and $T \in BL(X,Y)$ is bounded below, then $s_k(T) \geq \frac{1}{\|T^{-1}\|}$ for all $k \in \mathbb{N}$, and if $T \in BL(X,Y)$ is an isometry, then $s_k(T) = 1$ for all $k \in \mathbb{N}$.

Now we use Theorem 2.9 for showing approximation numbers of projections of norm 1 are either 1 or 0.

Corollary 2.13. *If $P \in BL(X)$ is a nonzero projection, then*

$$1 \leq s_k(P) \leq \|P\| \quad \forall\, k \leq \operatorname{rank}(P)$$

and $s_k(P) = 0$ for every $k > \operatorname{rank}(P)$. In particular, if $\|P\| = 1$, then

$$s_k(P) = \begin{cases} 1, & k \leq \operatorname{rank}(P), \\ 0, & k > \operatorname{rank}(P). \end{cases}$$

Proof. Since $Px = x$ for all $x \in R(P)$, Theorem 2.9 with $M = R(P)$ implies that $s_k(P) \geq 1$ for $k \leq \operatorname{rank}(P)$. Thus,

$$1 \leq s_k(P) \leq \|P\| \quad \forall\, k \leq \operatorname{rank}(P).$$

The particular case is obvious. □

Let $\alpha_i \in \mathbb{R}$ be such that $\alpha_i \geq \alpha_{i+1} \geq 0$ for all $i \in \mathbb{N}$. Let $D \in BL(\ell^p)$, $1 \leq p \leq \infty$ be the diagonal operator defined by

$$D(x_1, x_2, \ldots) = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad (x_1, x_2, \ldots) \in \ell^p.$$

We have seen in [14] that $s_k(D) = \alpha_k$ for all $k \in \mathbb{N}$. We prove a generalization of this result as a Corollary to Theorem 2.9, where $\{\alpha_n\}$ is assumed to be a sequence of complex numbers.

Corollary 2.14. *Let $D \in BL(\ell^p)$, $1 \leq p \leq \infty$, be the diagonal operator defined by*

$$Dx = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

where $\alpha_n \in \mathbb{C}$ satisfy $|\alpha_n| \geq |\alpha_{n+1}|$, $n \in \mathbb{N}$. Then

$$s_k(D) = |\alpha_k| \quad \forall\, k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$ and $M = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$, where $e_i = (\delta_{in})$ with δ_{in}, $i \in \mathbb{N}$. Then $\|Dx\| \geq |\alpha_k| \|x\|$ for all $x \in M$. Hence, by Theorem 2.9, $s_k(D) \geq |\alpha_k|$. Taking $P_{k-1} \in BL(\ell^p)$ as

$$P_{k-1}x = (x_1, x_2, \ldots, x_{k-1}, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

we have

$$s_k(D) \leq \|D - P_{k-1}D\| = |\alpha_k|.$$

Thus $s_k(D) = |\alpha_k|$ for all $k \in \mathbb{N}$. □

The next corollary gives a relation between certain approximation numbers and norms of certain generalized inverses. We refer to [1] for theory and applications of generalized inverses. In particular, recall that for $T \in BL(X,Y)$, $S \in BL(Y,X)$ is called a *{1}- inverse of* T if $TST = T$.

Corollary 2.15. *Let $T \in BL(X,Y)$ be such that there exists $S \in BL(Y,X)$ satisfying $TST = T$. Then*

$$s_k(T) \geq \frac{1}{\|S\|} \quad \forall\, k \leq rank(T).$$

In particular, if X and Y are Hilbert spaces and T has closed range, then

$$s_k(T) \geq \frac{1}{\|T^\dagger\|} \quad \forall\, k \leq \mathrm{rank}(T),$$

where $T^\dagger$ is the Moore-Penrose generalized inverse of T.

Proof. Since $TST = T$, ST is a projection and $\mathrm{rank}(ST) = \mathrm{rank}(T)$. Therefore, by Corollary 2.13,

$$1 \leq s_k(ST) \leq s_k(T)\|S\| \quad \forall\, k \leq \mathrm{rank}(T).$$

The particular case follows by noticing that $TT^\dagger T = T$, whenever $R(T)$ is closed. □

Remark 2.16. We may observe that Corollary 2.11 is a particular case of Corollary 2.15 .

Corollary 2.17. *Let $T \in BL(X)$ and λ be an eigenvalue of T. Then*

$$s_k(T) \geq |\lambda| \quad \forall\, k \leq dim(N(\lambda I - T)).$$

In particular, if $\dim(N(\lambda I - T)) = \infty$, then

$$s_k(T) \geq |\lambda| \quad \forall\, k \in \mathbb{N}.$$

Proof. Let $M_\lambda := N(\lambda I - T)$, and $k \in \mathbb{N}$ be such that $dim(M_\lambda) \geq k$. Then $\|Tx\| = |\lambda|\|x\|$ for all $x \in M_\lambda$ and hence by Theorem 2.9, $s_k(T) \geq |\lambda|$. □

Example 2.18. Let (α_n) be a bounded sequence in $\mathbb{C}$ and $\beta \in \mathbb{C}$. Let $D \in BL(\ell^p)$, $1 \leq p \leq \infty$, be defined by

$$Dx = (\beta x_1, \alpha_1 x_2, \beta x_3, \alpha_2 x_4, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Then β is an eigenvalue of infinite geometric multiplicity so that by Corollary 2.17, $s_k(D) \geq |\beta|$. If, in addition, $|\alpha_n| \leq |\beta|$ for all $n \in \mathbb{N}$, then we have $s_k(D) \leq \|D\| = |\beta|$ so that, in this case, $s_k(D) = |\beta|$ for all $k \in \mathbb{N}$.

If $T \in BL(X,Y)$ and M is a subspace of X, then we define

$$\nu_M(T) := \inf\{\|Tx\| : x \in M, \|x\| = 1\}.$$

Also, we shall denote by $\mathcal{M}_k(X)$ the set of all subspaces M of X such that $\dim(M) \geq k$. Then the quantity

$$\sup_{M \in \mathcal{M}_k(X)} \nu_M(T)$$

coincides with $u_k(T)$, the k^{th} Bernstein Number of T (See [15]). It is clear from Theorem 2.9 that

$$s_k(T) \geq \sup_{M \in \mathcal{M}_k(X)} \nu_M(T) = u_k(T). \tag{2.1}$$

Remark 2.19. Let X and Y be Hilbert spaces and $T \in BL(X,Y)$ be compact. Then we have $s_k(T) = \nu_M(T)$ for some subspace M of X of dimension k ([18], page 165). Hence in this case, if $s_k(T) \geq \alpha$ for some $\alpha > 0$, then there exists a subspace M of X such that $\dim(M) = k$ and $\|Tx\| \geq \alpha\|x\|$ for all $x \in M$. Hence in this special case, we have a positive answer to the question asked in Remark 2.10.

Now we show that the inequality in (2.1) can be strict for some operator. We first prove a lemma in this regard by modifying the arguments in the proof of Lemma 11.11.4 in [14]. In the following, card$K$ denotes the cardinality of a set K.

Lemma 2.20. *Let M be a subspace of ℓ^p, $1 \leq p < \infty$, such that dim$(M) \geq n$. Then there exists $e \in M$ such that $\|e\|_\infty = 1$ and card$\{k : |e(k)| = 1\} \geq n$.*

Proof. Let $1 \leq p < \infty$. Since $M \subset \ell^p \subset \ell^\infty$ is a finite dimensional subspace, the closed unit ball U_M of M (with respect to $\|\cdot\|_\infty$) is compact and convex in M. Hence by the Krein-Milman Theorem ([18]), there exists an extreme point e. Now since $e \in \ell^p$ also, $\|e\|_\infty = 1 = |e(k_0)|$ for some $k_0 \in \mathbb{N}$.

Let $K := \{k : |e(k)| = 1\}$. Assume that card$\{K\} < n$. Then, since $e \in \ell^p$, the number $\alpha := \sup\{|e(k)| : k \notin K\} < 1$. Now let

$$N = \{x \in M : x(k) = 0 \text{ for all } k \in K\}.$$

Then N is a nontrivial subspace of M. Let $u \neq 0$ be an element of U_N and $\delta := 1-\alpha$. We claim that $e \pm \delta u \in U_M$. To see this, note that $|e(k) \pm \delta u(k)| = 1$ for $k \in K$ and $|e(k) \pm \delta u(k)| \leq \alpha + \delta = 1$ for $k \notin K$. Hence

$$\|e \pm \delta u\|_\infty = \max\{|e(k) \pm \delta u(k)| : k \in \mathbb{N}\} \leq 1.$$

Thus $e \pm \delta u \in U_M$. But then e can not be an extreme point of U_M, and the assumption card$\{K\} < n$ can not be true. Thus card$\{k : |e(k)| = 1\} \geq n$. □

The following example shows that the inequality in 2.1 is strict for the inclusion operator $I : \ell^2 \to \ell^\infty$.

Example 2.21. Let $I : \ell^2 \to \ell^\infty$ be the natural injection. Then $\nu_M(I) \leq \frac{1}{\sqrt{2}}$ for all $M \in \mathcal{M}_2(\ell^2)$, and hence $s_2(I) \neq \sup\limits_{M \in \mathcal{M}_2(\ell^2)} \nu_M(I)$. To see this, note that, by Lemma 2.20, any subspace of ℓ^2 of dimension 2 contains an element e with $\|e\|_\infty = 1$ and card$\{k : |e(k)| = 1\} \geq 2$. Hence $\|e\|_2 \geq \sqrt{2}$. Then the element $u := \frac{e}{\|e\|_2}$ satisfies $\|u\|_2 = 1$ and $\|Iu\|_\infty \leq \frac{1}{\sqrt{2}}$. Therefore, $\nu_M(I) \leq \frac{1}{\sqrt{2}}$ for all $M \in \mathcal{M}_2(\ell^2)$, whereas $s_2(I) = 1$ ([7]).

It has been proved in [2] (See Proposition 9.2 in [2]) that if X is a finite dimensional space, say with $\dim(X) = n$, then for $T \in BL(X)$,

$$s_n(T) = \begin{cases} \dfrac{1}{\|T^{-1}\|} & \text{if } T \text{ is invertible} \\ 0 & \text{if } T \text{ is not invertible.} \end{cases}$$

This is a particular case of the following general result for any finite rank operator.

Theorem 2.22. *Let $T \in BL(X,Y)$ be a finite rank operator, say* rank$(T) = n$, *and let $S \in BL(R(T),X)$ satisfy $TST = T$. Then*

$$\frac{1}{\|S\|} \leq s_n(T) \leq \frac{\|ST\|}{\|S\|}.$$

In particular, if X and Y are Hilbert spaces, then $s_n(T) = \frac{1}{\|T^\dagger\|}$, where $T^\dagger$ is the Moore-Penrose inverse of T.

Proof. By Corollary 2.15, we have $s_n(T) \geq \frac{1}{\|S\|}$.

Now since $R(T)$ is finite dimensional, there exists $y \in R(T)$ such that $\|y\| = 1$ and $\|Sy\| = \|S\|$. By the Hahn Banach theorem, there exists $f \in X'$ such that $\|f\| = 1$ and $f(Sy) = \|Sy\| = \|S\|$. Define $P : X \to Y$ by

$$Pu = \frac{1}{\|S\|} f(STu)y, \quad u \in X.$$

Then $\|P\| \leq \frac{\|ST\|}{\|S\|}$. Let $F = T - P$. Note that $Fu = 0$ for all $u \in N(T)$. Hence $N(T) \subseteq N(F)$. Since $TSy = y$, we have

$$F(Sy) = TSy - \frac{1}{\|S\|} f(STSy)y = 0.$$

Thus $Sy \in N(F)$ but $TSy = y \neq 0$. Hence, $N(F) \supsetneq N(T)$. Thus, $rank(F) \leq n - 1$ so that

$$s_n(T) \leq \|T - F\| = \|P\| \leq \frac{\|ST\|}{\|S\|}.$$

If X and Y are Hilbert spaces, then we can take $S = T^\dagger$, and in that case $T^\dagger T$ is an orthogonal projection onto $N(T)^\perp$. Thus we obtain $\|ST\| = 1$, and consequently, $s_n(T) = 1/\|T^\dagger\|$. $\square$

For $T \in BL(X, Y)$ and $k \in \mathbb{N}$, we have $s_k(T) = \text{dist}(T, \mathcal{F}_k(X, Y))$. A question of interest is whether one can replace the set $\mathcal{F}_k(X, Y)$ by a smaller collection. In this regard we have the following proposition proved in [4] (Also see [14]).

Theorem 2.23. *([4], pages 67, 71) Let X and Y be Banach spaces, $T \in BL(X, Y)$ and $k \in \mathbb{N}$. Then we have the following.*

(i) *If X is a Hilbert space, then*
$$s_k(T) = \inf\{\|T - TP\| : P \in \mathcal{F}_k(X) \text{ is an orthogonal projection}\}.$$
(ii) *If Y is a Hilbert space, then*
$$s_k(T) = \inf\{\|T - PT\| : P \in \mathcal{F}_k(Y) \text{ is an orthogonal projection}\}.$$

In view of Theorem 2.23, and also to obtain further estimates for $s_k(T)$, we introduce a few more quantities.

Definition 2.24. For $T \in BL(X, Y)$ and $k \in \mathbb{N}$, we define

$$\gamma_k(T) := \inf\{\|T - PT\| : P \in \mathcal{F}_k(Y) \text{ is a projection}\}.$$

Also, for a finite dimensional subspace M of X, we define

$$\eta(M, X) := \inf\{\|I - P\| : P \in BL(X) \text{ is a projection with } R(P) = M\},$$

and for $n \in \mathbb{N}$, we define

$$\widehat{\eta}_n(X) := \sup\{\eta(M, X) : M \text{ subspace of } X \text{ with } \dim(M) = n\}.$$

Note that $\{\gamma_k(T)\}$ is a nonincreasing sequence and $s_k(T) \leq \gamma_k(T)$ for all $k \in \mathbb{N}$.

Clearly, if X is a Hilbert space, then $\widehat{\eta}_k(X) = 1$ for all $k \in \mathbb{N}$. For a general normed linear space X, it is known ([14], Page 386) that if $\check{\eta}_k(X)$ is the quantity defined by

$$\check{\eta}_k(X) := \sup_{M \in \widehat{\mathcal{N}}_k(X)} \inf\{\|P\| : P \text{ is a projection on } X \text{ with } R(P) = M\},$$

then $\check{\eta}_k(X) \leq \sqrt{k}$ so that

$$\widehat{\eta}_k(X) \leq 1 + \check{\eta}_k(X) \leq 1 + \sqrt{k}.$$

For a real Banach space X, it was shown in [11] that

$$\check{\eta}_k(X) \leq \frac{2 + (k-1)\sqrt{k+2}}{k+1} \leq \sqrt{k}.$$

This leads to an improved estimate for $\widehat{\eta}_k(X)$, for a real Banach space X, namely,

$$\widehat{\eta}_k(X) \leq 1 + \frac{2 + (k-1)\sqrt{k+2}}{k+1} \leq 1 + \sqrt{k}.$$

In terms of $\widehat{\eta}_k$, we give a general relation between $s_k(T)$ and $\gamma_k(T)$ in the following.

Proposition 2.25. *Let $T \in BL(X,Y)$ and $k \in \mathbb{N}$. Then*

$$s_k(T) \leq \gamma_k(T) \leq \widehat{\eta}_{k-1}(Y) s_k(T) \quad \forall\, k \in \mathbb{N}.$$

Proof. Clearly $s_k(T) \leq \gamma_k(T)$ for all $k \in \mathbb{N}$. Now let $\epsilon > 0$ be given. Let $F \in \mathcal{F}_k(X,Y)$ be such that $\|T - F\| \leq s_k(T) + \epsilon$. Then there exists a projection $P \in BL(Y)$ with $R(P) = R(F)$ and $\|I - P\| \leq \widehat{\eta}_{k-1}(Y)$. Hence

$$\begin{aligned} \|T - PT\| = \|(I-P)T\| &\leq \|(I-P)(T-F)\| \\ &\leq \|I-P\|\,\|T-F\| \\ &\leq \widehat{\eta}_{k-1}(Y)(s_k(T) + \epsilon). \end{aligned}$$

Thus $s_k(T) \leq \gamma_k(T) \leq \widehat{\eta}_{k-1}(Y) s_k(T)$. □

Remark 2.26. Since for a Hilbert space X, $\widehat{\eta}_k(X) = 1$ for all $k \in \mathbb{N}$, we get Theorem 2.23(ii) as a corollary to Proposition 2.25.

Remark 2.27. Let X and Y be Banach spaces and $T \in BL(X,Y)$. We may recall that the essential norm of T, denoted by $\|T\|_{ess}$, is defined by

$$\|T\|_{ess} := \inf\{\|T - K\| : K \in BL(X,Y) \text{ is compact}\}.$$

Since $\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq \|T\|_{ess}$, it follows that for those operators T for which $\|T\| = \|T\|_{ess}$, we have

$$s_k(T) = \|T\| \quad \forall\, k \in \mathbb{N}.$$

In particular, for Toeplitz operators in $BL(\ell^p)$ with $1 < p < \infty$, $s_k(T) = \|T\|$ for all $k \in \mathbb{N}$, as for such operators we have $\|T\| = \|T\|_{ess}$ ([2, 3]). Of course, there are operators other than Toeplitz operators which satisfy $\|T\| = \|T\|_{ess}$. For example, consider $X = C[a,b]$ with $\|\cdot\|_\infty$ and $A : X \to X$ defined by

$$(Ax)(t) = tx(t), \quad t \in [a,b].$$

Then it can be seen that $\|A\| = \|A\|_{ess}$.

We also infer that if $T \in BL(X,Y)$ satisfies $\|T\| = \|T\|_{ess}$ and if X is separable, Y is a dual space of some separable normed linear space and $\{T_n\}$ and $\{\tilde{T}_n\}$ are as in Corollary 2.5, then $\lim_{n\to\infty} s_k(\tilde{T}_n) = \lim_{n\to\infty} s_k(T_n) = s_k(T) = \|T\|$.

Now, let X be a Banach space and for $k \in \mathbb{N}$, let

$$\mathcal{A}_k := \{A \in BL(X) : A + F \text{ is not invertible for any } F \in \mathcal{F}_k(X)\}.$$

In the following, we give estimates for approximation numbers of operators of the form $\lambda I - A$ for $A \in \mathcal{A}_k$.

Proposition 2.28. *Let X be a Banach space, $\lambda \in \mathbb{C}$ and $A \in \mathcal{A}_k$ for some $k \in \mathbb{N}$. Then*

$$s_k(\lambda I - A) \geq |\lambda|.$$

In particular, if $A \in \cap_{k=1}^{\infty} \mathcal{A}_k$ and $\lambda \in \mathbb{C}$, then

$$s_k(\lambda I - A) \geq |\lambda| \quad \forall\, k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$, $A \in \mathcal{A}_k$ and $F \in \mathcal{F}_k(X)$. Since $0 \in \sigma(A+F)$, for any $\lambda \in \mathbb{C}$, we have (See [12], Theorem 10.10(i))

$$\|(\lambda I - A) - F\| \geq r_\sigma(\lambda I - A - F) \geq |\lambda|.$$

Thus $s_k(\lambda I - A) \geq |\lambda|$. The remaining part of the theorem is obvious. □

Let X and Y be Banach spaces and $T \in BL(X,Y)$. We recall that T is said to be a *Fredholm operator* if $R(T)$ is closed and $dim(N(T))$ and $codim(R(T))$ are finite, and in that case, the *index* of T is defined as the number

$$\operatorname{ind}(T) := \dim(N(T)) - \operatorname{codim}(R(T)).$$

We may recall that an operator $T \in BL(X,Y)$ is a Fredholm operator of index zero if and only if there exists a finite rank operator F such that $T+F$ is invertible ([6], page 191). Hence, it follows that

$T \in BL(X)$ is a Fredholm operator of index zero if and only if there exists $k \in \mathbb{N}$ such that $T \notin \mathcal{A}_k$.

Thus, as a consequence of Proposition 2.28, if $T \in BL(X)$ is not a Fredholm operator of index zero, then for every $\lambda \in \mathbb{C}$,

$$s_k(\lambda I - T) \geq |\lambda| \quad \forall\, k \in \mathbb{N}.$$

3. Closed range operators and approximation numbers

Let X and Y be Hilbert spaces and $T \in BL(X,Y)$. If the operator T is compact, then the set of all nonzero singular values of T coincides with the set of square roots of nonzero elements of $\sigma(T^*T)$. It was shown in [8] that

$R(T)$ is closed if and only if 0 is not an accumulation point of $\sigma(T^*T)$.

A question of interest is whether it is possible to study the closedness of $R(T)$ using approximation numbers of T when X and Y are not necessarily Hilbert spaces. Again it is worthwhile recalling here that the closedness of range is connected with boundedness of a generalized inverse. It is known that if X, Y are Banach spaces and $T \in BL(X,Y)$ has a bounded $\{1\}$- inverse, then $R(T)$ is closed. Also if X, Y are Hilbert spaces, then the Moore-Penrose inverse $T^\dagger$ is bounded if and only if $R(T)$ is closed.

Suppose X and Y are Banach spaces and $T \in BL(X,Y)$ is such that $\lim\limits_{k\to\infty} s_k(T) = 0$. Then we know that T is a compact operator, and in that case $R(T)$ is not closed whenever T is of infinite rank.

But the converse need not be true. That is, if T has non-closed range, then it is not necessary that $\lim\limits_{k\to\infty} s_k(T) = 0$. To see this, consider the diagonal operator $D \in BL(\ell^2)$ defined by

$$Dx = (2x_1, \frac{1}{2}x_2, 2x_3, \frac{1}{3}x_4, 2x_5, \ldots), \quad x = (x_1, x_2, x_3, \ldots) \in \ell^2.$$

In this case, $R(D)$ is not closed since 0 is an accumulation point of the spectrum of D^*D (cf. [8]). We observe that $s_k(D) = 2$ for all $k \in \mathbb{N}$ (taking $\beta = 2$ and $\alpha_n = \frac{1}{n}$, $n \in \mathbb{N}$, in Example 2.18. However, we have $s_n(P_nDP_n) = \frac{1}{n}$ for all $n \geq 2$, so that $s_n(P_nDP_n) \to 0$ as $n \to \infty$, where $\{P_n\}$ is the sequence of projection operators in $BL(\ell^2)$ defined by

$$P_nx = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^2.$$

In fact, the above result is true in a more general setting.

Proposition 3.1. *Let X be a Banach space, $T \in BL(X)$ and for each $n \in \mathbb{N}$, let $P_n \in \mathcal{F}_n(X)$ be projections such that $P_n \to I$ pointwise. If $R(T)$ is not closed, then $\lim\limits_{n\to\infty} s_n(P_nTP_n) = 0$.*

Proof. For $n \in \mathbb{N}$, let $T_n := P_nTP_n$ and $\tilde{T}_n := T_n\,|_{R(P_n)}\colon R(P_n) \to R(P_n)$. Assume that $\{s_n(T_n)\}$ does not converge to 0. Then there exists a $d > 0$ and a subsequence $\{s_{n_k}(T_{n_k})\}$ of $\{s_n(T_n)\}$ such that $s_{n_k}(T_{n_k}) \geq d$ for all $k \in \mathbb{N}$. Hence

$$s_{n_k}(\tilde{T}_{n_k}) \geq \frac{1}{M} s_{n_k}(T_{n_k}) \geq \frac{d}{M} > 0,$$

where $M \geq 1$ is such that $\|P_n\| \leq M$ for all $n \in \mathbb{N}$ (See Proposition 2.4). Then $\tilde{T}_{n_k}$ are invertible and $s_{n_k}(\tilde{T}_{n_k}) = 1/\|\tilde{T}_{n_k}^{-1}\|$ for each $k \in \mathbb{N}$. In particular, $\|\tilde{T}_{n_k}^{-1}\| \leq \frac{M}{d}$. Hence for $x \in X$,

$$\|P_{n_k}x\| \;=\; \|\tilde{T}_{n_k}^{-1}\tilde{T}_{n_k}P_{n_k}x\| \leq \|\tilde{T}_{n_k}^{-1}\|\|\tilde{T}_{n_k}P_{n_k}x\| \leq \frac{M}{d}\|\tilde{T}_{n_k}P_{n_k}x\|$$

Letting $k \to \infty$, we have $\|x\| \leq \dfrac{M}{d}\|Tx\|$ for all $x \in X$. Thus, T is bounded below and hence $R(T)$ closed. Thus $s_n(T_n) \to 0$ as $n \to \infty$, if $R(T)$ is not closed. □

The above result generalizes the last part of the following result proved in [16].

Theorem 3.2. *(cf. [16]) Let $T \in BL(\ell^p)$, $1 < p < \infty$, and for $n \in \mathbb{N}$, let $P_n \in BL(\ell^p)$ be defined by*

$$P_nx = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Let $\{T_n\}$ be a sequence of operators in $BL(R(P_n))$ such that $T_nP_n \to T$ pointwise as $n \to \infty$. If $R(T)$ is not closed, then for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} s_{n-k+1}(T_n) = 0.$$

In particular, if $R(T)$ is not closed, then $\lim\limits_{n\to\infty} s_n(P_nTP_n) = 0$.

We would like to mention that the converse of Theorem 3.2 need not be true in general, i.e., $s_{n-k+1}(T_n) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$ does not imply that $R(T)$ is not closed. To see this, consider the following example.

Example 3.3. For $n \in \mathbb{N}$, let $T_n \in BL(\ell^2)$ be defined by

$$T_n x = (x_1, x_2, \ldots, \frac{x_m}{n}, \frac{x_{m+1}}{n}, \ldots, \frac{x_n}{n}, 0, 0, \ldots),\ x = (x_1, x_2, \ldots) \in \ell^2,$$

where $m = [\frac{n}{2}]$, the greatest integer less than or equal to $\frac{n}{2}$. For $n \in \mathbb{N}$, let $\tilde{T}_n := T_n\, |_{R(P_n)}: R(P_n) \to R(P_n)$, where P_n is as in Theorem 3.2. Then for each $k \in \mathbb{N}$, we have $s_{n-k+1}(\tilde{T}_n) = \frac{1}{n}$ for all sufficiently large n and hence $s_{n-k+1}(\tilde{T}_n) \to 0$ as $n \to \infty$, whereas the pointwise limit of $\tilde{T}_n P_n$ is the identity operator, which has closed range.

References

[1] Ben-Israel, Adi and Greville, Thomas N. E., *Generalized inverses*, (3rd edition) Springer-Verlag, New York, 2003.

[2] Böttcher, A., On the approximation numbers of large Toeplitz matrices, *Doc. Math.* **2** (1997), 1–29.

[3] Böttcher A., and Silbermann, B., *Analysis of Toeplitz operators*, Second edition, Springer, Berlin, 2006.

[4] Carl, B., and Stephani, I., Entropy, compactness and the approximation of operators. Cambridge Tracts in Mathematics, 98. Cambridge University Press, Cambridge, 1990.

[5] Deepesh, K. P., Kulkarni, S. H. and Nair, M. T., Approximation numbers of operators on normed linear spaces, *Integral Equations and operator theory*, **65** (2009), 529-542.

[6] Gohberg, I., Goldberg S., and Kaashoek, M. A., *Classes of linear operators. Vol. I.* Operator Theory: Advances and Applications, 49. Birkhuser Verlag, Basel, 1990.

[7] Hutton, C. V., Morrell, J. S., Retherford and J. R. Diagonal operators, approximation numbers, and Kolmogoroff diameters. *J. Approximation Theory* **16** (1976), no. 1, 48–80.

[8] Kulkarni, S. H., and Nair, M. T., A characterization of closed range operators. *Indian J. Pure Appl. Math.* 31 (2000), no. 4, 353–361.

[9] Lomakina, E. N., Estimates for the approximation numbers of a class of integral operators. I. (Russian) *Sibirsk. Mat. Zh.* **44** (2003), no. 1, 178–192; translation in *Siberian Math. J.* **44** (2003), no. 1, 147–159.

[10] Lomakina, E. N., Estimates for the approximation numbers of a class of integral operators. II. (Russian) *Sibirsk. Mat. Zh.* **44** (2003), no. 2, 372–388; translation in *Siberian Math. J.* **44** (2003), no. 2, 298–310.

[11] Makai, E., Jr., and Martini, H., Projections of normed linear spaces with closed subspaces of finite codimension as kernels. *Period. Math. Hungar.* **52** (2006), no. 1, 41–46.

[12] Nair, M., T., *Functional Analysis: A First Course*, Prentice-Hall of India, New Delhi, 2002.

[13] Pietsch, Albrecht, *Eigenvalues and s-numbers*, Cambridge studies in advanced mathematics 13, 1985.

[14] Pietsch, A., *Operator ideals*, Translated from German by the author, North-Holland, Amsterdam, 1980.

[15] Pietsch, A., s-numbers of operators in Banach spaces. *Studia Math.* **51** (1974), 201–223.

[16] Rogozhin, A., and Silbermann, B., On the approximation numbers for the finite sections of block Toeplitz matrices, Bull. London Math. Soc. **38** (2006), no. 2, 301–313.

[17] Schock, E., On projection methods for linear equations of the second kind. *J. Math. Anal. Appl.*, **45** (1974), 293–299.

[18] Sunder, V. S., *Functional analysis Spectral theory*, Hindustan Book Agency, New Delhi, 1997.

Department of Mathematics, IIT Madras, Chennai 600 036, India
E-mail address: deepesh.kp@gmail.com

Department of Mathematics, IIT Madras, Chennai 600 036, India
E-mail address: shk@iitm.ac.in

Department of Mathematics, IIT Madras, Chennai 600 036, India
E-mail address: mtnair@iitm.ac.in