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An Analogue of the Spectral Mapping Theorem for Condition Spectrum

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Abstract. For $0 < \epsilon < 1$, the ϵ-condition spectrum of an element a in a complex unital Banach algebra A is defined as,

$$\sigma_\epsilon(a) = \left\{ \lambda \in \mathbb{C} : \lambda - a \text{ is not invertible or } \|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon} \right\}.$$

This is a generalization of the idea of spectrum introduced in [5]. This is expected to be useful in dealing with operator equations. In this paper we prove a mapping theorem for condition spectrum, extending an earlier result in [5]. Let f be an analytic function in an open set Ω containing $\sigma_\epsilon(a)$. We study the relations between the sets $\sigma_\epsilon(\tilde{f}(a))$ and $f(\sigma_\epsilon(a))$. In general these two sets are different. We define functions $\phi(\epsilon), \psi(\epsilon)$ (that take small values for small values of ϵ) and prove that $f(\sigma_\epsilon(a)) \subseteq \sigma_{\phi(\epsilon)}(\tilde{f}(a))$ and $\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\epsilon)}(a))$. The classical Spectral Mapping Theorem is shown as a special case of this result. We give estimates for these functions in some special cases and finally illustrate the results by numerical computations.

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1. Introduction

The Spectral Mapping Theorem is a fundamental result in functional analysis of great importance. Let A be a complex algebra with unit 1. We shall identify $\lambda.1$ with λ. We recall that the spectrum of an element $a \in A$ is defined as

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} : \lambda - a \notin A^{-1} \right\},$$

where A^{-1} is the set of all invertible elements of A [9]. The Spectral Mapping Theorem says that if f is an analytic function on an open set containing $\sigma(a)$, then

$$f(\sigma(a)) = \sigma(\tilde{f}(a)).$$

There are several generalizations of the concept of the spectrum in literature such as Ransford spectrum [8], pseudospectrum [11], n-pseudospectrum [2, 3], condition spectrum [5] etc. It is natural to ask whether there are any results similar to the Spectral Mapping Theorem for these sets. It is known that similar results hold if f is an affine function, that is, $f(z) = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$. (see Theorem 2.7 [5], Theorem 2.2 [6]). However it is not true, if f is an arbitrary analytic function (see Example 1). In [6], the author gives an analogue of the Spectral Mapping Theorem for pseudospectrum in the matrix algebra. The author carries forward this work in his recent paper [7]. The aim of this paper is to obtain an analogue of the Spectral Mapping Theorem for condition spectra of elements in a Banach algebra. We begin with the definition of condition spectrum.

Definition 1.1. (ϵ-condition spectrum) Let A be a complex unital Banach algebra with unit 1 and $0 < \epsilon < 1$. The *ϵ-condition spectrum* of an element $a \in A$, denoted by $\sigma_\epsilon(a)$, is defined as,

$$\sigma_\epsilon(a) = \left\{ \lambda \in \mathbb{C} : \ \|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon} \right\}$$

with the convention that $\|\lambda - a\| \|(\lambda - a)^{-1}\| = \infty$, if $\lambda - a$ is not invertible. Note that because of this convention $\sigma(a) \subseteq \sigma_\epsilon(a)$.

Suppose X is a Banach space and $T : X \to X$ is a bounded linear map. Then $\lambda \notin \sigma_\epsilon(T)$ means that the operator equation $Tx - \lambda x = y$ has a stable solution for every $y \in X$. This fact makes the ϵ-condition spectrum a potentially useful tool in the numerical solutions of operator equations. See [5] for examples and elementary properties of the condition spectrum.

Let f be an analytic function on some open set Ω containing $\sigma_\epsilon(a)$. Since $\sigma(a) \subseteq \sigma_\epsilon(a) \subseteq \Omega$, $\tilde{f}(a)$ can be defined by functional calculus as,

$$\tilde{f}(a) = \frac{1}{2\pi i} \int_\Gamma f(z)(z - a)^{-1} dz,$$

where Γ is any contour that surrounds $\sigma(a)$ in Ω [9]. If f is a polynomial, then $\tilde{f}(a) = f(a)$([9], Theorem 10.25). In view of this, some authors use the notation $f(a)$ in place of $\tilde{f}(a)$. We use the notation $\tilde{f}$ as in [9]. Our aim is to study the relations between the sets $f(\sigma_\epsilon(a))$ and $\sigma_\epsilon(\tilde{f}(a))$. Note that, in general we can not expect $f(\sigma_\epsilon(a)) = \sigma_\epsilon(\tilde{f}(a))$ (see Example 1 below). In other words, the verbatim analogue of the Spectral Mapping Theorem is not true. Hence we define functions ϕ, ψ such that $\lim\limits_{\epsilon \to 0} \phi(\epsilon) = 0 = \lim\limits_{\epsilon \to 0} \psi(\epsilon)$ and prove that $f(\sigma_\epsilon(a)) \subseteq \sigma_{\phi(\epsilon)}(\tilde{f}(a))$ and $\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\epsilon)}(a))$. These functions ϕ and ψ depend on f and a. If for some f, $\phi(\epsilon) = \epsilon = \psi(\epsilon)$, then we would get $f(\sigma_\epsilon(a)) = \sigma_\epsilon(\tilde{f}(a))$ for that f. This happens when f is an affine function.

The following is an outline of the paper. In Section 2, the general theorem in the form of two set inclusions is stated and proved (Theorem 2.1). It is shown that the set inclusions reduce to an equality if the mapping is an affine function

(Remark 2.4). It is also shown that the usual Spectral Mapping Theorem as well as the pseudospectral mapping theorem of Lui [6] are special cases of our result (Remark 2.7). In Section 3, a weak version of the theorem is proved in a Banach algebra with some additional property (Theorem 3.4). In Section 4, we present some numerical experiments which illustrate the theory developed in the earlier sections.

2. Main theorem

First we give an example to show that $f(\sigma_\epsilon(a)) \neq \sigma_\epsilon(\tilde{f}(a))$ in general. Next we give an analogue of the Spectral Mapping Theorem for condition spectrum for complex analytic functions. The theorem is an easy consequence of the definition of the functions defined in the statement of the theorem.

Example 1. Let $A = \mathbb{C}^{2\times 2}$, the algebra of all 2×2 matrices with the operator norm $\|\cdot\|_2$. Let

$$P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $f(z) = z^2$, then

$$\tilde{f}(P) = P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $\sigma_\epsilon(\tilde{f}(P)) = \{1\}$. On the other hand $\sigma_\epsilon(P)$ contains complex numbers different from -1 and 1 (see Corollary 3.4, [5]). Hence $f(\sigma_\epsilon(P))$ contains complex numbers different from 1.

Theorem 2.1. *Let A be a complex Banach algebra with unit 1. For $a \in A$, $0 < \epsilon < 1$ sufficiently small, Ω a bounded open subset of $\mathbb{C}$ containing $\sigma_\epsilon(a)$ and f an analytic function on Ω, define*

$$\phi(\epsilon) = \sup_{\lambda \in \sigma_\epsilon(a)} \left\{ \frac{1}{\|f(\lambda) - \tilde{f}(a)\| \|[f(\lambda) - \tilde{f}(a)]^{-1}\|} \right\}.$$

If $\tilde{f}(a)$ is not a scalar multiple of unit, then $\phi(\epsilon)$ is well defined, $0 \le \phi(\epsilon) \le 1$, $\lim\limits_{\epsilon \to 0} \phi(\epsilon) = 0$ and for ϵ satisfying $\phi(\epsilon) < 1$, we have

$$f(\sigma_\epsilon(a)) \subseteq \sigma_{\phi(\epsilon)}(\tilde{f}(a)).$$

Further suppose f is injective on Ω and there exists ϵ_0 with $0 < \epsilon_0 < 1$ such that $\sigma_{\epsilon_0}(\tilde{f}(a)) \subseteq f(\Omega)$. For $0 < \epsilon \le \epsilon_0$ define

$$\psi(\epsilon) = \sup_{\mu \in f^{-1}(\sigma_\epsilon(\tilde{f}(a)))} \left\{ \frac{1}{\|\mu - a\| \|(\mu - a)^{-1}\|} \right\}.$$

Then $\psi(\epsilon)$ is well defined, $0 \le \psi(\epsilon) \le 1$, $\lim\limits_{\epsilon \to 0} \psi(\epsilon) = 0$ and for ϵ satisfying $\psi(\epsilon) < 1$, we have

$$\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\epsilon)}(a)).$$

Proof. First, we show that for each $a \in A$, $\phi(\epsilon)$ is well defined. Define $g:\mathbb{C}\to\mathbb{R}$ by,

$$g(\lambda) = \frac{1}{\|f(\lambda) - \tilde{f}(a)\|\|[f(\lambda) - \tilde{f}(a)]^{-1}\|}.$$

We claim that g is continuous. Clearly g is continuous on $\mathbb{C} \setminus \sigma(a)$. Let $\lambda \in \sigma(a)$. Then by the Spectral Mapping Theorem,

$$f(\lambda) \in f(\sigma(a)) = \sigma(\tilde{f}(a)).$$

Thus by our convention $g(\lambda) = 0$. To complete the proof of the claim, we need to show the following. If $\lambda_n \in \mathbb{C} \setminus \sigma(a)$, $\lambda_n \to \lambda \in \sigma(a)$, then $g(\lambda_n) \to 0$. Let $\{\lambda_n\}$ be such a sequence. Then $f(\lambda_n) - \tilde{f}(a) \to f(\lambda) - \tilde{f}(a)$. Hence $\{f(\lambda_n) - \tilde{f}(a)\}$ is a bounded sequence. On the other hand, since $f(\lambda) \in \sigma(\tilde{f}(a))$, $\|(f(\lambda_n) - \tilde{f}(a))^{-1}\| \to \infty$ (Lemma 10.17 of [9]). Hence $g(\lambda_n) \to 0$. This proves the claim. Next for $0 < \epsilon < 1$, $\sigma_\epsilon(a)$ is a compact set [5] and $\phi(\epsilon) = \sup\{g(\lambda) : \lambda \in \sigma_\epsilon(a)\}$. Hence $\phi(\epsilon)$ is well defined, that is, finite.

Next we prove $\lim\limits_{\epsilon\to 0} \phi(\epsilon) = 0$. Let $\epsilon_n > 0$ be a sequence converging to 0. By compactness of $\sigma_{\epsilon_n}(a)$ there exist $\lambda_n \in \sigma_{\epsilon_n}(a)$ such that $g(\lambda_n) = \phi(\epsilon_n)$. Now λ_n is a bounded sequence and hence has a convergent subsequence $\{\lambda_{n_k}\}$ converging to λ. Hence $\{\lambda_{n_k} - a\}$ is a bounded sequence. On the other hand, $\|\lambda_{n_k} - a\|\|(\lambda_{n_k} - a)^{-1}\| \geq \frac{1}{\epsilon_{n_k}}$ for all n_k. Thus $\|(\lambda_{n_k} - a)^{-1}\| \to \infty$ as $n_k \to \infty$. This imply that $\lambda - a$ is not invertible. Thus $\lambda \in \sigma(a)$ and $f(\lambda) \in \sigma(\tilde{f}(a))$. Now $\{f(\lambda_{n_k}) - \tilde{f}(a)\}$ converges to $f(\lambda) - \tilde{f}(a)$. Hence $\{f(\lambda_{n_k}) - \tilde{f}(a)\}$ is bounded and $\|(f(\lambda_{n_k}) - \tilde{f}(a))^{-1}\| \to \infty$. This gives $\phi(\epsilon_{n_k}) = g(\lambda_{n_k}) \to 0$. Since $\phi(\epsilon_n)$ is monotonically increasing $\phi(\epsilon_n) \to 0$. Now let ϵ be sufficiently small so that $0 \leq \phi(\epsilon) < 1$ and let $\lambda \in \sigma_\epsilon(a)$. Then $g(\lambda) \leq \phi(\epsilon)$. Hence

$$\|f(\lambda) - \tilde{f}(a)\|\|[f(\lambda) - \tilde{f}(a)]^{-1}\| = \frac{1}{g(\lambda)} \geq \frac{1}{\phi(\epsilon)}.$$

This means that $f(\lambda) \in \sigma_{\phi(\epsilon)}(\tilde{f}(a))$. Thus

$$f(\sigma_\epsilon(a)) \subseteq \sigma_{\phi(\epsilon)}(\tilde{f}(a)).$$

Next we assume that f is injective on Ω and there exists ϵ_0 with $0 < \epsilon_0 < 1$ such that $\sigma_{\epsilon_0}(\tilde{f}(a)) \subseteq f(\Omega)$ and we show that for each $a \in A$ and $0 < \epsilon \leq \epsilon_0$, $\psi(\epsilon)$ is well defined. Define $h : \mathbb{C} \to \mathbb{R}$ by,

$$h(\mu) = \frac{1}{\|\mu - a\|\|(\mu - a)^{-1}\|}$$

We claim that h is continuous. Clearly h is continuous on $\mathbb{C} \setminus \sigma(a)$. Let $\mu \in \sigma(a)$, by our convention $h(\mu) = 0$. To complete the proof of the claim we need to show the following. If $\mu_n \in \mathbb{C} \setminus \sigma(a)$, $\mu_n \to \mu \in \sigma(a)$, then $h(\mu_n) \to 0$. Let $\{\mu_n\}$ be such a sequence. Then $\mu_n - a \to \mu - a$. Hence $\{\mu_n - a\}$ is a bounded sequence. On the other hand, since $\mu \in \sigma(a)$, $\|(\mu_n - a)^{-1}\| \to \infty$ (Lemma 10.17 of [9]). Hence $h(\mu_n) \to 0$. This proves the claim. Since $h(\mu) \leq 1$ for all $\mu \in \mathbb{C}$, $\psi(\epsilon)$ is well defined and $0 \leq \psi(\epsilon) \leq 1$.

Next we prove $\lim_{\epsilon\to 0}\psi(\epsilon) = 0$. Let $\epsilon_n > 0$ be a sequence converging to 0. Since

$$\psi(\epsilon_n) = \sup_{\mu\in f^{-1}(\sigma_{\epsilon_n}(\tilde{f}(a)))} h(\mu),$$

and $f^{-1}(\sigma_{\epsilon_n}(\tilde{f}(a)))$ is closed and bounded, hence compact, there exists $\mu_n \in f^{-1}(\sigma_{\epsilon_n}(\tilde{f}(a)))$ such that $\psi(\epsilon_n) = h(\mu_n)$. Since each $\mu_n \in \Omega$, which is bounded, it has a convergent subsequence $\{\mu_{n_k}\}$ converging to μ. On the other hand, since $f(\mu_{n_k}) \in \sigma_{\epsilon_{n_k}}(\tilde{f}(a))$, we have

$$\|f(\mu_{n_k}) - \tilde{f}(a)\|\|[f(\mu_{n_k}) - \tilde{f}(a)]^{-1}\| \geq \frac{1}{\epsilon_{n_k}}$$

for all n_k. Thus $\|[f(\mu_{n_k}) - \tilde{f}(a)]^{-1}\| \to \infty$ as $n_k \to \infty$. This implies that $f(\mu) - \tilde{f}(a)$ is not invertible. Thus $f(\mu) \in \sigma(\tilde{f}(a))$. Since f is injective $\mu \in \sigma(a)$ and $h(\mu) = 0$. Since h is continuous $\psi(\epsilon_{n_k}) = h(\mu_{n_k}) \to h(\mu) = 0$. Finally since ψ is monotonically increasing $\psi(\epsilon_n) \to 0$.

Now let ϵ be sufficiently small so that $0 \leq \psi(\epsilon) < 1$. Let $\lambda \in \sigma_\epsilon(\tilde{f}(a)) \subseteq \sigma_{\epsilon_0}(\tilde{f}(a)) \subseteq f(\Omega)$. Consider $\mu \in \Omega$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\sigma_\epsilon(\tilde{f}(a)))$, hence $h(\mu) \leq \psi(\epsilon)$, that is,

$$\|\mu - a\|\|(\mu - a)^{-1}\| \geq \frac{1}{\psi(\epsilon)}.$$

Thus $\mu \in \sigma_{\psi(\epsilon)}(a)$. Hence $\lambda = f(\mu) \in f(\sigma_{\psi(\epsilon)}(a))$. This proves

$$\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\epsilon)}(a)). \qquad \square$$

Remark 2.2. Combining the two inclusions, we get

$$f(\sigma_\epsilon(a)) \subseteq \sigma_{\phi(\epsilon)}(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\phi(\epsilon))}(a)).$$

and

$$\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\epsilon)}(a)) \subseteq \sigma_{\phi(\psi(\epsilon))}(\tilde{f}(a)).$$

Remark 2.3. Since for every $a \in A$, $\lim_{\epsilon\to 0}\phi(\epsilon) = 0 = \lim_{\epsilon\to 0}\psi(\epsilon)$, $\sigma(a) = \bigcap_{0<\epsilon<1}\sigma_\epsilon(a)$ and ϕ, ψ are monotonically increasing functions, the usual Spectral Mapping Theorem can be deduced from Theorem 2.1. However, it may be noted that the proof of Theorem 2.1 uses the Spectral Mapping Theorem.

Remark 2.4. Let $a \in A$ and $f(z) = \alpha + \beta z$ where α, β are complex numbers with $\beta \neq 0$. Then

$$\begin{aligned}\phi(\epsilon) &= \sup_{\lambda\in\sigma_\epsilon(a)} \frac{1}{\|\beta\lambda - \beta a\|\|(\beta\lambda - \beta a)^{-1}\|} \\ &= \sup_{\lambda\in\sigma_\epsilon(a)} \frac{1}{\|\lambda - a\|\|(\lambda - a)^{-1}\|} \\ &= \epsilon\end{aligned}$$

In a similar way we have $\psi(\epsilon) = \epsilon$. Thus $\sigma_\epsilon(\alpha + \beta a) = \alpha + \beta\sigma_\epsilon(a)$ (see (7) of Theorem 2.7 in [5]), that is $\sigma_\epsilon(\tilde{f}(a)) = f(\sigma_\epsilon(a))$. This leads to the following question.

Question 2.5. Let f be a non-constant analytic function defined on a nonempty open set Ω in the complex plane. Suppose

$$f(\sigma_\epsilon(a)) = \sigma_{\phi(\epsilon)}(\tilde{f}(a))$$

for all $a \in A$ with $\sigma(a) \subset \Omega$. Then does it follow that $\phi(\epsilon) = \epsilon$ and $f(z) = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$?

Remark 2.6. The hypothesis that $\tilde{f}(a)$ is not a scalar multiple of unity cannot be dropped from Theorem 2.1. Let f and P be as in Example 1. Since $\tilde{f}(P) = I$, we have $\sigma_{\phi(\epsilon)}(\tilde{f}(P)) = \{1\}$, On the other hand we have noted in Example 1 that $\sigma_\epsilon(P)$ contains complex numbers different from -1 and 1. Hence $f(\sigma_\epsilon(P))$ contains complex numbers different from 1. Thus

$$f(\sigma_\epsilon(P)) \nsubseteq \sigma_{\phi(\epsilon)}(\tilde{f}(P)).$$

Remark 2.7. Let $\Lambda_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq 1/\epsilon\right\}$ denote the pseudospectrum of a. (See [11] for examples and applications of pseudospectrum.) It was shown in [4] that if a is not a scalar multiple of 1, then there exist positive numbers α, β depending on a, such that $\sigma_\epsilon(a) \subseteq \Lambda_{\alpha\epsilon}(a)$ and $\Lambda_\epsilon(a) \subseteq \sigma_{\beta\epsilon}(a)$. (See [4] for exact values of α, β.) Now from Theorem 2.1

$$f(\Lambda_\epsilon(a)) \subseteq f(\sigma_{\beta\epsilon}(a)) \subseteq \sigma_{\phi(\beta\epsilon)}(\tilde{f}(a)) \subseteq \Lambda_{\alpha\phi(\beta\epsilon)}(\tilde{f}(a)).$$
$$\Lambda_\epsilon(\tilde{f}(a)) \subseteq \sigma_{\beta\epsilon}(\tilde{f}(a)) \subseteq f(\sigma_{\psi(\beta\epsilon)}(a)) \subseteq f(\Lambda_{\alpha\psi(\beta\epsilon)}(a)).$$

This is a more general form of the pseudospectral mapping theorem given in [6].

3. Weak versions

The functions ϕ and ψ defined in the last section are continuous and monotonically increasing but it appears to be difficult to find the values of these functions explicitly in a Banach algebra. In this section, we replace these functions ϕ, ψ with the functions $\gamma_\epsilon, \delta_\epsilon$ respectively that are relatively easier to estimate. The results using these functions are weaker in the following sense

1. We need to assume some additional property for Banach algebras.
2. We need to take a bigger neighborhood Ω.

The next lemma describes this additional property.

Lemma 3.1. *Let A be a complex unital Banach algebra with the following property:*

$$\forall \ a \in A^{-1}, \ \exists \ b \in A \setminus A^{-1} \quad \textit{such that} \quad \|a - b\| = \frac{1}{\|a^{-1}\|} \tag{3.1}$$

Then for every $a \in A$ such that a is not a scalar multiple of unity and $\lambda \in \sigma_\epsilon(a)$, there exists an element $b \in A$ such that

$$\lambda \in \sigma(a+b) \quad \textit{with} \quad \|b\| \leq \epsilon\|\lambda - a\|.$$

Proof. We refer to [5] for a proof of this result. □

The article [5] contains examples of Banach algebras satisfying Property 3.1. In particular the uniform algebras and matrix algebras satisfy this property (see Examples 2.18, 2.20 in [5]).

Lemma 3.2. *Let A be a complex Banach algebra with unit 1. Let $0 < \epsilon < 1$ and $a \in A$ be such that a is not scalar multiple of unit. Let $m = \inf\{\|z.1 - a\| : z \in \mathbb{C}\}$. Then*

$$\bigcup_{\|b\| \leq m\epsilon} \sigma(a+b) \subseteq \sigma_\epsilon(a).$$

Further if A has Property 3.1 stated in Lemma 3.1 *then*

$$\sigma_\epsilon(a) \subseteq \bigcup_{\|b\| \leq \frac{2\epsilon}{1-\epsilon}\|a\|} \sigma(a+b).$$

Thus for such algebras

$$\bigcup_{\|b\| \leq m\epsilon} \sigma(a+b) \subseteq \sigma_\epsilon(a) \subseteq \bigcup_{\|b\| \leq \frac{2\epsilon}{1-\epsilon}\|a\|} \sigma(a+b).$$

Proof. Let $\lambda \in \sigma(a+b)$ with $b \in A$ and $\|b\| \leq \epsilon m$. Since

$$m = \inf\{\|z.1 - a\| : z \in \mathbb{C}\} \leq \|\lambda - a\|,$$

we have $\|b\| \leq \epsilon\|\lambda - a\|$. Hence by Theorem 2.16 of [5], we obtain

$$\sigma(a+b) \subseteq \sigma_\epsilon(a).$$

Next suppose A has Property 3.1 mentioned in Lemma 3.1. Let $\lambda \in \sigma_\epsilon(a)$. Then by Theorem 2.9 of [5], $|\lambda| \leq \dfrac{1+\epsilon}{1-\epsilon}\|a\|$.
Also by Lemma 3.1, $\lambda \in \sigma(a+b)$ for some $b \in A$ with $\|b\| \leq \epsilon\|\lambda - a\|$. Now

$$\|b\| \leq \epsilon\|\lambda - a\| \leq \epsilon(|\lambda| + \|a\|) \leq \frac{2\epsilon}{1-\epsilon}\|a\|.$$

This proves the second relation. □

Theorem 3.3. *Let A be a complex Banach algebra with unit 1 satisfying Property 3.1 stated in* Lemma 3.1. *Let $a \in A$ and Ω be an open set containing $\sigma(a)$. Then there exist $0 < \epsilon < 1$ such that $\sigma_\epsilon(a) \subseteq \Omega$.*

Proof. Recall that the map $a \longmapsto \sigma(a)$ is upper semicontinuous [1]. Hence there exist $\delta > 0$ such that $\sigma(a+b) \subseteq \Omega$ for all $b \in A$ with $\|b\| \leq \delta$ (see Theorem 10.20 of [9]). Now take $\epsilon = \dfrac{\delta}{\delta + 2\|a\|}$. Lemma 3.2 gives $\sigma_\epsilon(a) \subseteq \Omega$. □

The following theorem is the weak version of Theorem 2.1

Theorem 3.4. *Let A be a complex Banach algebra with unit* 1 *satisfying Property* 3.1 *mentioned in* Lemma 3.1. *Let $a \in A$, $0 < \epsilon < 1$ sufficiently small, Ω be an open subset of $\mathbb{C}$ containing* $\bigcup_{\|b\| \leq \frac{2\epsilon}{1-\epsilon}\|a\|} \sigma(a+b)$. *Let f be an injective analytic function defined on Ω. Assume that $a, \tilde{f}(a)$ are not scalar multiples of unity. Define*

$$\gamma_\epsilon := \sup\Big\{\|\tilde{f}(a+p) - \tilde{f}(a)\| : \|p\| \leq \frac{2\epsilon}{1-\epsilon}\|a\|\Big\}.$$
$$m := \inf\{\|z.1 - a\| : z \in \mathbb{C}\} > 0.$$
$$\delta_\epsilon := \sup\Big\{\|q\| : \|\tilde{f}(a+q) - \tilde{f}(a)\| \leq \frac{2\epsilon}{1-\epsilon}\|\tilde{f}(a)\|\Big\}.$$
$$m' := \inf\big\{\|z.1 - \tilde{f}(a)\| : z \in \mathbb{C}\big\} > 0.$$

Then $\lim_{\epsilon \to 0} \gamma_\epsilon = 0 = \lim_{\epsilon \to 0} \delta_\epsilon$.

1. Let $\epsilon > 0$ be such that $\frac{\gamma_\epsilon}{m'} < 1$. Then $f(\sigma_\epsilon(a)) \subseteq \sigma_{\frac{\gamma_\epsilon}{m'}}(\tilde{f}(a))$.
2. Let $\epsilon > 0$ be such that $\frac{\delta_\epsilon}{m} < 1$. Then $\sigma_\epsilon(\tilde{f}(a)) \subseteq f(\sigma_{\frac{\delta_\epsilon}{m}}(a))$.

Proof. Since the map $x \longmapsto \tilde{f}(x)$ is continuous, we obtain $\lim_{\epsilon \to 0} \gamma_\epsilon = 0$. Let $g : f(\Omega) \to \Omega$ be the inverse of f. Using the continuity of the map $y \longmapsto \tilde{g}(y)$, we obtain $\lim_{\epsilon \to 0} \gamma_\epsilon = 0$. Next let $\epsilon > 0$ be such that $\frac{\gamma_\epsilon}{m'} < 1$ and let $\lambda \in \sigma_\epsilon(a)$. By Lemma 3.2 there exist $b \in A$ with $\|b\| \leq \dfrac{2\epsilon}{1-\epsilon}\|a\|$ such that $\lambda \in \sigma(a+b)$. Then by the Spectral Mapping Theorem, $f(\lambda) \in \sigma(\tilde{f}(a+b))$. Let $c = \tilde{f}(a+b) - \tilde{f}(a)$, then $\|c\| \leq \gamma_\epsilon$ and by the above lemma,

$$f(\lambda) \in \sigma(\tilde{f}(a) + c) \subseteq \sigma_{\frac{\gamma_\epsilon}{m'}}(\tilde{f}(a)).$$

This proves 1.

Let $\lambda \in \sigma_\epsilon(\tilde{f}(a))$. Then by Lemma 3.2, $\lambda \in \sigma(\tilde{f}(a) + d)$ for some $d \in A$ with $\|d\| \leq \dfrac{2\epsilon}{1-\epsilon}\|\tilde{f}(a)\|$. By the inverse mapping theorem, [9], there exist $p \in A$ and $\epsilon_1 > 0$ such that $\|p\| \leq \epsilon_1$ and $\tilde{f}(a+p) = \tilde{f}(a) + d$. Thus by the Spectral Mapping Theorem there exist $\mu \in \sigma(a+p)$ such that,

$$f(\mu) = \lambda \in \sigma(\tilde{f}(a+p)) = \sigma(\tilde{f}(a) + d).$$

Claim: $\mu \in \sigma_{\frac{\delta_\epsilon}{m}}(a)$.

$$\|d\| = \|\tilde{f}(a+p) - \tilde{f}(a)\| \leq \frac{2\epsilon}{1-\epsilon}\|\tilde{f}(a)\|.$$

Hence

$$\|p\| \leq \delta_\epsilon := \sup\Big\{\|q\| : \|\tilde{f}(a+q) - \tilde{f}(a)\| \leq \frac{2\epsilon}{1-\epsilon}\|\tilde{f}(a)\|\Big\}.$$

Now by Lemma 3.2, $\mu\in\sigma_{\frac{\delta_\epsilon}{m}}(a)$. This proves the claim. Hence $\lambda=f(\mu)\in f(\sigma_{\frac{\delta_\epsilon}{m}}(a))$. This proves 2. $\square$

Remark 3.5. If $\tilde{f}$ has a bounded Fréchet derivative in a neighborhood Ω containing $\sigma_\epsilon(a)$, then γ_ϵ can be estimated as follows. Let A be a complex unital Banach algebra, $a \in A$ and $0 < \epsilon < 1$. Let $(D\tilde{f})_x$ denote the Fréchet derivative of $\tilde{f}$ at $x \in A$. Let

$$L_\epsilon := \sup\left\{\|(D\tilde{f})_x\| : \quad x \in A, \quad \|x-a\| \leq \frac{2\epsilon}{1-\epsilon}\|a\|\right\}.$$

Then, for $b \in A$ with $\|b\| \leq \dfrac{2\epsilon}{1-\epsilon}\|a\|$, we have by the Mean Value Theorem [10],

$$\|\tilde{f}(a+b)-\tilde{f}(a)\| \leq L_\epsilon\|b\| \leq \frac{2\epsilon}{1-\epsilon}L_\epsilon\|a\|.$$

Thus

$$\gamma_\epsilon \leq \frac{2\epsilon}{1-\epsilon}L_\epsilon\|a\|.$$

Remark 3.6. Let A be a complex unital Banach algebra, $a \in A$ and $0 < \epsilon < 1$. Let f be an injective analytic function defined on an open set Ω containing $\sigma_\epsilon(a)$. Let $g : f(\Omega) \to \Omega$ be the inverse of f. If $\tilde{g}$ has a bounded Fréchet derivative in a neighborhood of $\sigma_\epsilon(\tilde{f}(a))$, then δ_ϵ can be estimated as follows. Let

$$L'_\epsilon := \sup\left\{\|(D\tilde{g})_x\| : \quad x \in A, \quad \|x-\tilde{f}(a)\| \leq \frac{2\epsilon}{1-\epsilon}\|\tilde{f}(a)\|\right\}.$$

Then, for $d' \in A$ with $\|d'-\tilde{f}(a)\| \leq \dfrac{2\epsilon}{1-\epsilon}\|a\|$, we have by the Mean Value Theorem [10],

$$\|\tilde{g}(d')-\tilde{g}(\tilde{f}(a))\| \leq L'_\epsilon\|d'-\tilde{f}(a)\| \leq \frac{2\epsilon}{1-\epsilon}L'_\epsilon\|\tilde{f}(a)\|.$$

Thus

$$\delta_\epsilon \leq \frac{2\epsilon}{1-\epsilon}L'_\epsilon\|\tilde{f}(a)\|.$$

In the following examples we give estimates for $\gamma_\epsilon, \delta_\epsilon$ for the functions $f(z) = z^2$, $f(z) = z^3$ and $f(z) = e^z$.

Example 2. Let $A = (C[1,2], \|\cdot\|_\infty)$, $0 < \epsilon < 1$ sufficiently small and $a \in A$ is defined by $a(x) = x$ for all $x \in [1,2]$. Then $\|a\|_\infty = 2$, $\sigma(a) = [1,2]$.

$$\begin{aligned} m &:= \inf\{\|z-a\|_\infty : z \in \mathbb{C}\}.\\ &= \inf\{\sup\{|z-x| : x \in [1,2]\} : z \in \mathbb{C}\}.\\ &= \inf\{\max\{|z-1|, |z-2|\} : z \in \mathbb{C}\}.\\ &= \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} m' &:= \inf\{\|z - a^2\|_\infty : z \in \mathbb{C}\}. \\ &= \inf\{\sup\{|z - x^2| : x \in [1,2]\} : z \in \mathbb{C}\}. \\ &= \inf\{\max\{|z-1|, |z-4|\} : z \in \mathbb{C}\}. \\ &= \frac{3}{2}. \\ \gamma_\epsilon &:= \sup\left\{\|(a+p)^2 - a^2\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}. \\ &= \sup\left\{\|2ap + p^2\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}. \\ &\leq \left\{4\|p\|_\infty + \|p\|_\infty^2 : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\} \leq \frac{16\epsilon}{(1-\epsilon)^2}. \end{aligned}$$

Hence $\sigma_\epsilon(a)^2 \subseteq \sigma_{\epsilon_1}(a^2)$, (by Theorem 3.4). Where $\epsilon_1 = \frac{32\epsilon}{3(1-\epsilon)^2}$

$$\begin{aligned} \delta_\epsilon &:= \sup\left\{\|q\|_\infty : \|(a+q)^2 - a^2\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}. \\ &= \sup\left\{\|q\|_\infty : \|2qa + q^2\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}. \\ &= \sup\left\{\|q\|_\infty : \|2q + q^2\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}. \\ &\leq \sup\left\{\|q\|_\infty : 2\|q\|_\infty - \|q\|_\infty^2 \leq \frac{8\epsilon}{1-\epsilon}\right\}. \\ &\leq 1 - \sqrt{1-9\epsilon}. \end{aligned}$$

This gives $\sigma_\epsilon(a^2) \subseteq \sigma_{\epsilon_1}(a)^2$, where $\epsilon_1 = 2(1 - \sqrt{1-9\epsilon})$.

Next for $f(z) = z^3$.

$$\begin{aligned} m' &:= \inf\{\|z - a^3\|_\infty : z \in \mathbb{C}\}. \\ &= \inf\{\sup\{|z - x^3| : x \in [1,2]\} : z \in \mathbb{C}\}. \\ &= \inf\{\max\{|z-1|, |z-8|\} : z \in \mathbb{C}\}. \\ &= \frac{7}{2}. \\ \gamma_\epsilon &:= \sup\left\{\|(a+p)^3 - a^3\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}. \\ &= \sup\left\{\|3a^2p + 3ap^2 + p^3\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}. \\ &\leq \left\{3\|a\|_\infty^2\|p\|_\infty + 3\|a\|\|p\|_\infty^2 + \|p\|_\infty^3 : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}. \\ &\leq \frac{16\epsilon(3 - 3\epsilon - 2\epsilon^2)}{(1-\epsilon)^3}. \end{aligned}$$

Hence $\sigma_\epsilon(a)^3 \subseteq \sigma_{\epsilon_1}(a^3)$ (by Theorem 3.4). Where $\epsilon_1 = \frac{32\epsilon(3-3\epsilon-2\epsilon^2)}{7(1-\epsilon)^3}$

$$\begin{aligned}
\delta_\epsilon :=& \sup\left\{\|q\|_\infty : \|(a+q)^3 - a^3\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}.\\
=& \sup\left\{\|q\|_\infty : \|3qa^2 + 3q^2a + q^3\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}.\\
=& \sup\left\{\|q\|_\infty : \|3q + 3q^2 + q^3\|_\infty \leq \frac{8\epsilon}{1-\epsilon}\right\}.\\
\leq& \sup\left\{\|q\|_\infty : 3\|q\|_\infty - 3\|q\|_\infty^2 - \|q\|_\infty^3 \leq \frac{8\epsilon}{1-\epsilon}\right\}.\\
\leq& 8\epsilon.
\end{aligned}$$

This gives $\sigma_\epsilon(a^3) \subseteq \sigma_{16\epsilon}(a)^3$.

For $f(z) = e^z$

$$\begin{aligned}
m' :=& \inf\{\|z - \exp(a)\|_\infty : z \in \mathbb{C}\}.\\
=& \inf\{\sup\{|z - \exp(x)| : x \in [1,2]\} : z \in \mathbb{C}\}.\\
=& \inf\{\max\{|z-e|, |z-e^2|\} : z \in \mathbb{C}\}.\\
=& \frac{e(e-1)}{2}.
\end{aligned}$$

$$\begin{aligned}
\gamma_\epsilon :=& \sup\left\{\|\exp(a+p) - \exp(a)\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}.\\
=& \sup\left\{\|\exp(a)(\exp(p)-1)\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}.\\
\leq& \left\{e^{\|a\|_\infty}\|\exp(p) - 1\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}.\\
\leq& \left\{e^2\|\sum_{k=1}^{\infty}\frac{p^k}{k!}\|_\infty : \|p\|_\infty \leq \frac{4\epsilon}{1-\epsilon}\right\}.\\
\leq& e^2(e^{\frac{4\epsilon}{1-\epsilon}} - 1).
\end{aligned}$$

Hence $\exp(\sigma_\epsilon(a)) \subseteq \sigma_{\epsilon_1}(\exp(a))$ (by Theorem 3.4). Where $\epsilon_1 = \dfrac{2e(e^{4\epsilon/1-\epsilon} - 1)}{e-1}$.

$$\delta_\epsilon := \sup\left\{\|q\|_\infty : \|\exp(a+q) - \exp(a)\|_\infty \leq \frac{2e^2\epsilon}{1-\epsilon}\right\}.$$

Let $\exp(a) = c, \exp(a+q) = d = c+b$. Then

$$\begin{aligned}
\delta_\epsilon =& \sup\left\{\|\log(c+b) - \log(c)\|_\infty : \|b\|_\infty \leq \frac{2e^2\epsilon}{1-\epsilon}\right\}.\\
=& \sup\left\{\|\log(1 + c^{-1}b)\|_\infty : \|b\|_\infty \leq \frac{2e^2\epsilon}{1-\epsilon}\right\}.
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup\left\{\log(1+\|c^{-1}\|\|b\|_\infty) : \|b\|_\infty \leq \frac{2e^2\epsilon}{1-\epsilon}\right\}.\\
&\leq \log\left(1+\|c^{-1}\|\frac{2e^2\epsilon}{1-\epsilon}\right)\\
&\leq \log\left(1+\frac{2e^3\epsilon}{1-\epsilon}\right).
\end{aligned}
$$

This gives $\sigma_\epsilon(\exp(a)) \subseteq \exp(\sigma_{\epsilon_1}(a))$. Where $\epsilon_1 = 2\log\left(1+\frac{2e^3\epsilon}{1-\epsilon}\right)$.

Example 3. Let $A = BL(l^\infty, \|\cdot\|)$, $0 < \epsilon < 1$ sufficiently small and $T \in A$ is defined by $T(x)(i) = x(i+1)$ for all $x \in l^\infty$, the left shift operator.

Consider $f(z) = z^3$. From Example 2.14 of [5] we have,

$$\sigma_\epsilon(T) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\right\}.$$

From Theorem 2.1 we have,

$$\phi(\epsilon) = \sup_{\lambda\in\sigma_\epsilon(T)} g(\lambda), \quad \text{where } g(\lambda) = \frac{1}{\|\lambda^3 - T^3\|\|(\lambda^3-T^3)^{-1}\|}.$$

Also it is well known that $\sigma(T) = \{\lambda : |\lambda| \leq 1\}$ [9]. Hence $g(\lambda) = 0$ for $|\lambda| \leq 1$. Next let $1 < |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}$. Then

$$\|\lambda^3 - T^3\| = 1 + |\lambda|^3, \|(\lambda^3 - T^3)^{-1}\| = \frac{1}{|\lambda|^3 - 1}.$$

Hence,

$$
\begin{aligned}
g(\lambda) &= \frac{1}{\|\lambda^3 - T^3\|\|(\lambda^3-T^3)^{-1}\|} = \frac{|\lambda|^3-1}{|\lambda|^3+1}\\
&\leq \frac{(\frac{1+\epsilon}{1-\epsilon})^3 - 1}{2} = \frac{6\epsilon + 2\epsilon^3}{2(1-\epsilon)^3} = \frac{\epsilon(3+\epsilon^2)}{(1-\epsilon)^3}
\end{aligned}
$$

Thus $\phi(\epsilon) \leq \frac{\epsilon(3+\epsilon^2)}{(1-\epsilon)^3}$. Note that,

$$
\begin{aligned}
\sigma_\epsilon(T)^3 &= \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{(1+\epsilon)^3}{(1-\epsilon)^3}\right\} \subseteq \sigma_{\phi(\epsilon)}(T^3)\\
&\subseteq \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1+\phi(\epsilon)}{1-\phi(\epsilon)}\right\}\\
&\subseteq \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{(1-\epsilon)^3 + \epsilon(3+\epsilon^2)}{(1-\epsilon)^3 - \epsilon(3+\epsilon^2)}\right\}
\end{aligned}
$$

Next $\sigma_\epsilon(T^3) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\}$. From Theorem 2.1, we have $\psi(\epsilon) = \sup\{h(\mu) : \mu^3 \in \sigma_\epsilon(T^3)\}$, where $h(\mu) = \frac{1}{\|\mu - T\|\|(\mu-T)^{-1}\|}$. Since $\sigma(T^3) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, hence $h(\mu) = 0$ for $|\mu| \leq 1$. Consider $1 < |\mu^3| \leq \dfrac{1+\epsilon}{1-\epsilon}$.

$$\begin{aligned} h(\mu) &= \frac{1}{\|\mu - T\|\|(\mu - T)^{-1}\|} \leq \frac{|\mu| - 1}{\mu + 1} \\ &\leq \frac{(\frac{1+\epsilon}{1-\epsilon})^{1/3} - 1}{2} = \frac{1}{2}\left[\left(1 + \frac{2\epsilon}{1-\epsilon}\right)^{1/3} - 1\right] \leq \frac{\epsilon}{3(1-\epsilon)}. \end{aligned}$$

Thus $\psi(\epsilon) \leq \dfrac{\epsilon}{3(1-\epsilon)}$. Hence,

$$\begin{aligned} \sigma_\epsilon(T^3) &= \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\right\} \\ &\subseteq \sigma_{\psi(\epsilon)}(T)^3 \\ &\subseteq \left\{\lambda \in \mathbb{C} : |\lambda| \leq \left(\frac{1+\psi(\epsilon)}{1-\psi(\epsilon)}\right)^3\right\} \\ &\leq \left\{\lambda \in \mathbb{C} : |\lambda| \leq \left(\frac{3-2\epsilon}{3-4\epsilon}\right)^3\right\}. \end{aligned}$$

4. Numerical results

In this section, we report the results of some numerical experiments done using matlab.
Let $A = (C[1,2], \|\cdot\|_\infty)$, $\epsilon = 0.1$ and $a \in A$ be defined by $a(x) = x$ for all $x \in [1,2]$ as in Example 2. If $f(z) = z^2$. Then $\tilde{f}(a) = a^2$ defined by $a^2(x) = a(x)a(x) = x^2$. The ϵ-condition spectrum of a can be calculated as follows, Let $z = \alpha + \beta i$. Then there are four cases.

- $\alpha < 1$. In this case $\|z - a\|_\infty = \sqrt{(\alpha-2)^2 + \beta^2}$ and $\|(z-a)^{-1}\|_\infty = 1/\sqrt{(\alpha-1)^2 + \beta^2}$
- $1 \leq \alpha < 1.5$. In this case $\|z - a\|_\infty = \sqrt{(\alpha-2)^2 + \beta^2}$ and $\|(z-a)^{-1}\|_\infty = 1/|\beta|$
- $1.5 \leq \alpha < 2$. In this case $\|z - a\|_\infty = \sqrt{(\alpha-1)^2 + \beta^2}$ and $\|(z-a)^{-1}\|_\infty = 1/|\beta|$
- $\alpha \geq 2$. In this case $\|z - a\|_\infty = \sqrt{(\alpha-1)^2 + \beta^2}$ and $\|(z-a)^{-1}\|_\infty = 1/\sqrt{(\alpha-2)^2 + \beta^2}$

Thus ϵ-condition spectrum can be calculated explicitly using the definition. In a similar way ϵ-condition spectrum of $\tilde{f}(a)$ also can be calculated. To calculate approximate value of $\phi(\epsilon)$ we choose a certain number of uniformly distributed points in $\sigma_\epsilon(a)$, compute $\|z^2 - a^2\|_\infty\|(z^2 - a^2)^{-1}\|_\infty$ at each of these points and take the maximum of these values as an approximation of $\phi(\epsilon)$. Similarly $\psi(\epsilon)$ is computed. For $\epsilon = 0.1$, these computed values turn out to be $\phi(\epsilon) = 0.1332$ and

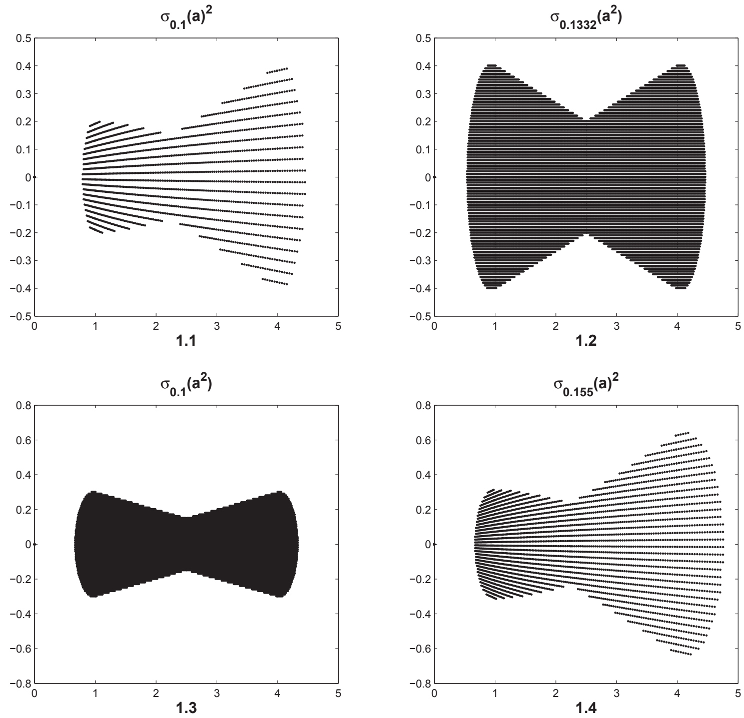

FIGURE 1

$\psi(\epsilon) = 0.155$. From Theorem 2.1 we have the following inclusions

$$\sigma_{0.1}(a)^2 \subseteq \sigma_{0.1332}(a^2).$$
$$\sigma_{0.1}(a^2) \subseteq \sigma_{0.155}(a)^2.$$

The following figures are obtained using matlab. Figure 1.1 shows $\sigma_{0.1}(a)^2$, Figure 1.2 shows $\sigma_{0.1332}(a^2)$, Figure 1.3 shows $\sigma_{0.1}(a^2)$, and Figure 1.4 shows $\sigma_{0.155}(a)^2$.

The condition spectrum of an $n \times n$ matrix T can be computed as follows. It is proved in [5] (Theorem 2.9) that,

$$|\lambda| \le \frac{1+\epsilon}{1-\epsilon}\|T\| \quad \text{for all} \quad \lambda \in \sigma_\epsilon(T).$$

We can consider certain number of uniformly distributed points in the disc

$$\left\{ z \in \mathbb{C} : |z| \le \frac{1+\epsilon}{1-\epsilon}\|T\| \right\},$$

evaluate $\|z - T\|\|(z-T)^{-1}\|$ at each of these points and include and save those z for which

$$\|z - T\|\|(z-T)^{-1}\| \ge \frac{1}{\epsilon}.$$

This gives $\sigma_\epsilon(T)$. We plot the points to the complex plane using matlab and obtain the figure for $\sigma_\epsilon(T)$. For each such chosen points z in $\sigma_\epsilon(T)$, we compute

$$\frac{1}{\|f(z)-\tilde{f}(T)\|\|(f(z)-\tilde{f}(T))^{-1}\|}$$

and take the maximum value as an approximation of $\phi(\epsilon)$ defined in Theorem 2.1. Similarly we calculate $\psi(\epsilon)$. As in the case of pseudospectrum [11], condition spectrum of a matrix also can be computed using different algorithms. Since our aim is only to illustrate our results, we have used a very basic algorithm. We do not make any claim about the efficiency of this algorithm.

We have considered $(\mathbb{C}^{10\times 10}, \|\cdot\|_2)$ and the following 10×10 Toeplitz matrix.

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ 0 & \dots & . & 0 & 1 & 1 \\ 0 & \dots & \dots & . & 0 & 1 \end{bmatrix}_{10\times 10}$$

(1) Let $f(z)=z^2$ and $\epsilon = 0.1$. Then $\tilde{f}(T)=T^2$ is also a Toeplitz matrix

$$T^2 = \begin{bmatrix} 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ 0 & \dots & . & 0 & 1 & 2 \\ 0 & \dots & \dots & . & 0 & 1 \end{bmatrix}_{10\times 10}$$

Using the algorithm explained above we obtain $\phi(\epsilon) = 0.1662$ and $\psi(\epsilon) = 0.1602$. From Theorem 2.1 we have the following inclusions

$$\sigma_{0.1}(T)^2 \subseteq \sigma_{0.1662}(T^2).$$
$$\sigma_{0.1}(T^2) \subseteq \sigma_{0.1602}(T)^2.$$

The figures obtained using matlab computations are given in Figure 2. Figure 2.1 shows $\sigma_{0.1}(T)^2$, Figure 2.2 shows $\sigma_{0.1662}(T^2)$, Figure 2.3 shows $\sigma_{0.1}(T^2)$, and Figure 2.4 shows $\sigma_{0.1602}(T)^2$.

(2) Let $f(z)=e^z$ and $\epsilon = 0.01$. Then $\tilde{f}(T)=\exp(T)$ is also a Toeplitz matrix.

$$\exp(T) = \begin{bmatrix} e & e & 1.3591 & 0.4530 & \dots & 0.000 \\ 0 & e & e & 1.3591 & \dots & 0.001 \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ 0 & \dots & . & 0 & e & e \\ 0 & \dots & \dots & . & 0 & e \end{bmatrix}_{10\times 10}$$

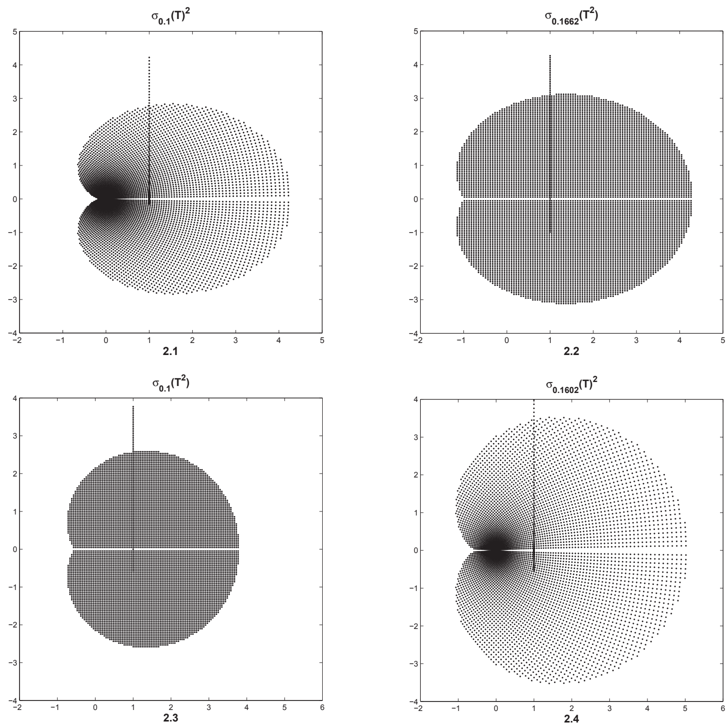

Figure 2

Using the same algorithm we obtain $\phi(\epsilon) = 0.0195$ and $\psi(\epsilon) = 0.0258$. Thus by Theorem 2.1, we have the following two inclusions

$$e^{\sigma_{0.01}(T)} \subseteq \sigma_{0.0195}(\exp(T)).$$
$$\sigma_{0.01}(\exp(T)) \subseteq e^{\sigma_{0.0258}(T)}.$$

The figures obtained using matlab computations are given in Figure 3. Figure 3.1 shows $e^{\sigma_{0.01}(T)}$, Figure 3.2 shows $\sigma_{0.0195}(\exp(T))$, Figure 3.3 shows $\sigma_{0.01}(\exp(T))$, and Figure 3.4 shows $e^{\sigma_{0.0258}(T)}$.

(3) In the next example we consider a random matrix J of order 3×3.

$$J = \begin{bmatrix} 0.5 & 1 & -1 \\ 1.5 & -0.5 & 0.25 \\ 0.75 & 1.5 & 1.25 \end{bmatrix}_{3\times 3}$$

Let $f(z) = z^3$ and $\epsilon = 0.01$, we have $\tilde{f}(J) = J^3$ is given by

$$J^3 = \begin{bmatrix} 0.125 & -3.5 & -0.25 \\ 2.6719 & 3.1562 & 1.703 \\ -2.3906 & -0.0938 & 2.8906 \end{bmatrix}_{3\times 3}$$

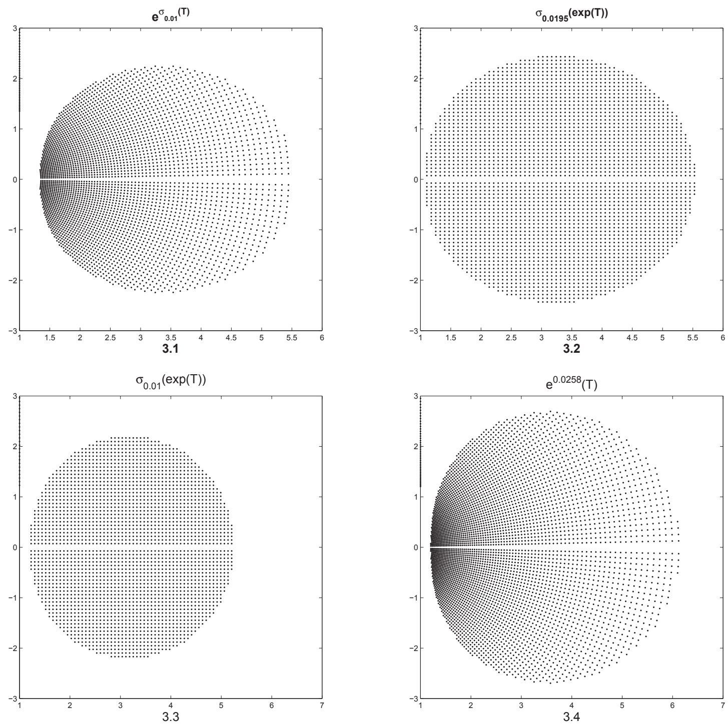

Figure 3

As above we obtain $\phi(\epsilon) = 0.0381$ and $\psi(\epsilon) = 0.1945$ Thus by Theorem 2.1, we have the following two inclusions

$$\sigma_{0.01}(J)^3 \subseteq \sigma_{0.0381}(J^3), \qquad \sigma_{0.01}(J^3) \subseteq \sigma_{0.1411}(J)^3.$$

The figures obtained using matlab computations are given in Figure 4. Figure 4.1 shows $\sigma_{0.01}(J)^3$, Figure 4.2 shows $\sigma_{0.0381}(J^3)$, Figure 4.3 shows $\sigma_{0.01}(J^3)$, and Figure 4.4 shows $\sigma_{0.1411}(J)^3$.

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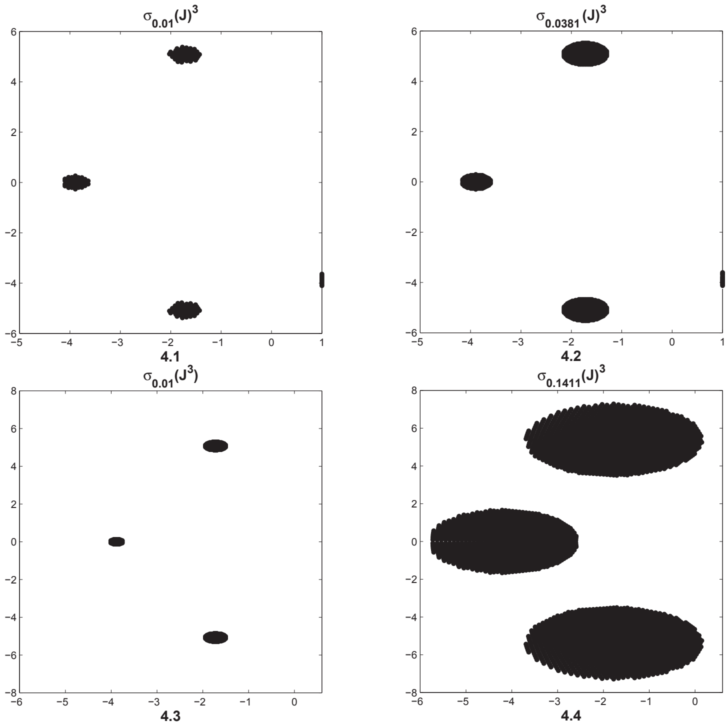

Figure 4

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